On variations of the action

Consider a harmonic oscillator,

\[ L = \frac{1}{2} \dot{x}^2 - \frac{1}{2} \omega^2 x^2. \]  

(1)

Consider paths with \( x(0) = x(T) = 0 \) where \( T = 2\pi/\omega \) is the period of the oscillator,

\[ S = \int_0^T L(x, \dot{x}) \, dt. \]  

(2)

Stationary paths are determined by

\[ \frac{\delta S}{\delta x(t)} = -d \frac{\partial L}{\partial \dot{x}} + \frac{\partial L}{\partial x} = 0. \]  

(3)

Inserting the Lagrangian, we find

\[ \frac{\delta S}{\delta x(t)} = -d \frac{\ddot{x}}{\dot{x}} - \omega^2 x, \]  

(4)

so the equation of motion Eq.(3) is

\[ \ddot{x} + \omega^2 x = 0, \]  

(5)

which is solved by

\[ x(t) = A \sin\omega t \]  

(6)

for any value of \( A \). This is a stationary point of the action and

\[ L = \frac{1}{2} A^2 \omega^2 \cos 2\omega t, \]  

(7)

giving

\[ S = \int_0^T L \, dt = 0. \]  

(8)

Now to see whether these solutions are minima or maxima or what, we return to Eq.(4) and differentiate again,

\[ \frac{\delta^2 S}{\delta x(t) \delta x(t')} = \frac{\delta}{\delta x(t')} \left[ -\ddot{x}(t) - \omega^2 x(t) \right] \]  

(9)

\[ = -\delta''(t - t') - \omega^2 \delta(t - t'). \]  

(10)

This matrix has both positive and negative eigenvalues, which means that \( \delta S \) can be either positive or negative, depending on the form of the variation \( \delta x(t) \) around the solution (6). To be explicit,

\[ S[x + \delta x] = S[x] + \int \frac{\delta S}{\delta x(t)} \delta x(t) \, dt + \frac{1}{2} \int \int \frac{\delta^2 S}{\delta x(t) \delta x(t')} \delta x(t) \delta x(t') \, dt \, dt' + \cdots. \]  

(11)
The first term $S[x]$ is zero as above, and the first variation vanishes by the Lagrange equation. When we use Eq. (10), the second variation is seen to be

$$\frac{1}{2} \int dt \delta x(t) \left[ -\delta \ddot{x}(t) - \omega^2 \delta x(t) \right].$$

(12)

Since $\delta x(t)$ must satisfy the boundary conditions, we can consider particular cases of the form $\delta x(t) = \epsilon \sin \frac{1}{2} n \omega t$. [These are eigenfunctions of the differential operator (10).] The second variation is then

$$\frac{1}{2} \left( \frac{1}{4} n^2 - 1 \right) \omega^2 \epsilon^2 \int dt [\delta x(t)]^2.$$  

(13)

This is negative for $n = 1$; it is zero for $n = 2$ (because it is a rescaling of the classical solution, and the action is independent of $A$); it is positive for $n > 2$.

Thus the classical solution (6) is a saddle point. Note that the reason that the $n = 1$ variation is unstable (i.e., a maximum at $\delta x = 0$) is that we chose the variable $T$ to be an entire period of the oscillator. If we had chosen it to be a half-period, then the classical solution would have been stable against all variations except that of rescaling $A$. If we choose $T$ to be some other integer multiple of a half-period, say $\pi m/\omega$, then all variations with $n < m$ would represent instabilities.

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