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FERMION PAIR PRODUCTION IN A STRONG ELECTRIC FIELD

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ABSTRACT. The initial-value problem for the quantum back-reaction in spinor QED is formulated and solved in the semiclassical mean field approximation, for a homogeneous but time-dependent electric field $E(t)$. We apply the method of adiabatic regularization to the Dirac equation in order to renormalize the expectation value of the current and derive a finite coupled set of ordinary differential equations for the time evolution of the system. We solve this system in (1+1) dimensions numerically and compare the solution to a simple model based on a relativistic Boltzmann-Vlasov equation, with a particle production source term inferred from the Schwinger particle creation rate and a Pauli-blocking factor. This model reproduces very well the time behavior of the electric field and the creation rate of e^+e^- pairs of the semiclassical calculation.

1 Introduction

The rate of creation of pairs of charged particles in a static and homogeneous external electric field was computed long ago [1–6]. This process has been used extensively in color-flux-tube models to describe multiparticle production in hadronic collisions [6–13]. A strong color-electric field is assumed to be formed between receding hadronic sources—quarks or hadrons—and quark-antiquark and gluon pairs emerge in the presence of the field by tunneling. In models for the production of a quark-gluon plasma the source for the field is the fragmenting nuclei.

In solving a dynamical problem with a strong initial electric field the effect of the produced particles on the electric field (the back-reaction) should be taken into consideration [11–13]. The construction of the usual models for quark-gluon plasma creation has often been based on oversimplifications. One is the modification of the Schwinger expression for the time independent rate of pair production so that it becomes time dependent through the time variation of the electric field (even though a fixed external electric field is inherent in Schwinger's derivation); moreover the transverse and longitudinal momentum distributions of the produced particles are chosen according to results of a WKB calculation [12] . It is not clear whether this choice is entirely appropriate.

In order to evaluate the dynamic rate of pair production self-consistently, we propose to study the quantum back-reaction of spin-$\frac{1}{2}$ fields in a spatially homogeneous classical electric field through the semiclassical Maxwell equations. The back-reaction problem for quantum fields has been developed particularly in the study of quantum fields in curved space [14–19]. In formulating the back-reaction problem, divergences appear in the expectation value of the conserved currents, and a renormalization procedure which guarantees well-defined finite equations is required. Adiabatic regularization is a useful approach which enables one to dispose of these infinities in a way that is consistent with conventional renormalization for a variety of spatially homogeneous problems [14–20], and is easy to implement in a practical numerical procedure. This method has been successfully applied to the study of the back-reaction problems in which the matter field is a scalar field [17–20].

In Ref. [20] we found that results based on calculations in semiclassical scalar electrodynamics are very similar to those obtained from a model based on a phenomenological relativistic Boltzmann-Vlasov equation. It is of interest to see if the same results obtain for fermions which, unlike bosons, possess no classical limit, and for which Pauli blocking, rather than Bose enhancement, implies a quite different consequence from the consideration of quantum statistics.

To apply adiabatic regularization to the case of fermions, we again express the Fourier components of the field operators in a WKB-like form. This enables us to isolate the ultraviolet divergences via an adiabatic expansion and to perform a mode-by-mode subtraction. In Section 2 we derive the coupled equations for the fields in the semiclassical limit of QED, and in Section 3 we discuss the adiabatic regularization procedure. In Section 4 we present our numerical results in (1+1) dimensions and compare them with the phenomenological Boltzmann-Vlasov model.

2 QED in the semiclassical mean field limit

The lagrangian density for electrodynamics is

$$\mathcal{L} = \bar{\psi} i\gamma^\mu (\partial_\mu + ieA_\mu)\psi - m\bar{\psi}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{2.1}$$

where the metric convention is taken as $(+ - --)$. For the γ matrices we use the convention of Bjorken and Drell [21]

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \qquad i = 1, 2, 3 \tag{2.2}$$

where I is the identity matrix and σ^i are the Pauli matrices.

In the mean field approximation we quantize only the Dirac field, while the electromagnetic field is treated classically. This approximation may be derived formally as the leading term in the large N limit of QED, where N is the number of charged matter fields [19]. The resulting coupled field equations read

$$(i\gamma^\mu \partial_\mu - e\gamma^\mu A_\mu - m)\psi(x) = 0 \tag{2.3}$$

$$\partial_\mu F^{\mu\nu} = \langle j^\nu \rangle = \frac{e}{2}\langle [\bar{\psi}, \gamma^\nu \psi] \rangle \tag{2.4}$$

where the expectation value is with respect to the initial state of the spinor field. The commutator in the electric current guarantees a zero expectation value for any charge conjugation eigenstate. Expressing the solution of the Dirac equation as

$$\psi(x) = (i\gamma^\mu \partial_\mu - e\gamma^\mu A_\mu + m)\phi(x), \tag{2.5}$$

and inserting (2.5) into (2.3), it follows that ϕ satisfies the quadratic Dirac equation

$$\left[(i\partial_\mu - eA_\mu)^2 - \frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu} - m^2\right]\phi(x) = 0, \tag{2.6}$$

where ϕ is a four component spinor. Here we consider the case where the electric field is spatially homogeneous so that the field strength $F^{\mu\nu}$ depends only on time. Owing to homogeneity the semiclassical Maxwell equations (2.4) allow only configurations where $\langle j^0 \rangle = 0$. We take the electric field to be in the direction of the z axis, and we choose a gauge such that only $A \equiv A^3(t)$ is nonvanishing. Then the second order Dirac equation becomes

$$\left[\Box + e^2A^2(t) + 2iA(t)\partial_3 - ie\partial_0 A(t)\gamma^0\gamma^3 + m^2\right]\phi(x) = 0. \tag{2.7}$$

Spatial homogeneity implies that there exist solutions of the form

$$\phi_{\mathbf{k}s}(x) = e^{i\mathbf{k}\cdot\mathbf{x}} f_{\mathbf{k}s}(t)\chi_s, \tag{2.8}$$

where

$$\chi_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \chi_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \tag{2.9}$$

$$\chi_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \quad \chi_4 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}. \tag{2.10}$$

These spinors are chosen to be eigenvectors of $\alpha^3 = \gamma^0\gamma^3$ in the representation (2.2) for the γ matrices. The eigenvalues of α^3 are $\lambda_s = 1$ for $s = 1, 2$ and $\lambda_s = -1$ for $s = 3, 4$. These spinors satisfy the normalization and completeness conditions,

$$\begin{aligned} \sum_{\alpha=1}^{4} (\chi_r^\dagger)^\alpha (\chi_s)_\alpha &= 2\delta_{rs} \\ \sum_{r=1}^{4} (\chi_r^\dagger)^\alpha (\chi_r)_\beta &= 2\delta^\alpha_{\ \beta}\ . \end{aligned} \tag{2.11}$$

Substituting (2.8) into (2.7) it follows that the mode functions $f_{\mathbf{k}s}(t)$ satisfy

$$\frac{d^2 f_{\mathbf{k}s}(t)}{dt^2} + \left[\omega_{\mathbf{k}}^2(t) - i\lambda_s e\frac{dA}{dt}\right] f_{\mathbf{k}s}(t) = 0, \tag{2.12}$$

with

$$\omega_{\mathbf{k}}^2(t) \equiv p_3^2 + \mathbf{k}_\perp^2 + m^2 \quad , \quad \mathbf{k}_\perp^2 \equiv k_1^2 + k_2^2, \quad p_i \equiv k^i - eA^i(t) \quad . \tag{2.13}$$

Equations (2.12) are second order differential equations, and therefore for each s there are two independent solutions. Let $f^+_{\mathbf{k}s}$ and $f^-_{\mathbf{k}s}$ be the two independent solutions of (2.12), which become positive- and negative-frequency solutions in the absence of the electric field. Clearly at the moment we have eight different solutions for the second-order equation (2.6), namely $f^\pm_{\mathbf{k}s}$ for $s = 1, 2, 3, 4$. However, the Dirac equation (2.3) has only four independent solutions . If we restrict ourselves to solutions which belong to the set $s = 1, 2$ or to the set $s = 3, 4$ we shall see that from each set one can construct a linearly independent set of solutions of the Dirac equation. The form introduced in (2.8) allows us to write ψ as

$$\psi_{\mathbf{k}s}(x) \equiv \hat{D}\phi_{\mathbf{k}s}(x) \equiv (i\gamma^0\partial_0 + \gamma^i k_i - e\gamma^3 A_3 + m)\phi_{\mathbf{k}s}(x). \tag{2.14}$$

Explicitly, the two sets of independent solutions of the Dirac equation may be taken to be

$$\psi^\pm_{\mathbf{k}s} = e^{i\mathbf{k}\cdot\mathbf{x}}\hat{D}f^\pm_{\mathbf{k}s}\chi_s, \quad s = 1, 2 \tag{2.15}$$

and

$$\psi^\pm_{\mathbf{k}s} = e^{i\mathbf{k}\cdot\mathbf{x}}\hat{D}f^\pm_{\mathbf{k}s}\chi_s, \quad s = 3, 4\ . \tag{2.16}$$

Using Eqs. (2.14)–(2.16) we find (for a given $\mathbf{k}$)

$$\begin{aligned} \psi_r^{\pm\dagger}\psi_s^\pm &= \chi_r{}^\dagger f_r^{*\pm}\hat{D}^\dagger\hat{D}f_s^\pm\chi_s \qquad (2.17)\\ &= 2\left\{\omega^2_{\mathbf{k}}(t)f_r^{*\pm}f_s^\pm + \dot{f}_r^{*\pm}\dot{f}_s^\pm - i\lambda_s p_3\left[f_r^{*\pm}\dot{f}_s^\pm - \dot{f}_r^{*\pm}f_s^\pm\right]\right\}\delta_{rs} \end{aligned}$$

where either $r, s = 1, 2$ or $r, s = 3, 4$. An exactly analogous formula may be derived for $\psi_r^{\pm\dagger}\psi_s^\mp$. By differentiating these expressions with respect to time and by using Eq. (2.12), it can be readily verified that these inner products are time-independent. As $t \to -\infty$ there is no interaction between the fermion field and the electromagnetic field, and we can choose two independent plane wave solutions for Eq. (2.12),

$$\lim_{t\to-\infty} f^\pm_{\mathbf{k}s} = c_s e^{\mp i\omega_{\mathbf{k}}t}, \tag{2.18}$$

where c_s are constants. Insertion of these free solutions in the relation for $\psi_r^{\pm\dagger}\psi_s^\mp$ yields immediately

$$\psi_r^{\pm\dagger}\psi_s^{\mp} = 0\ . \tag{2.19}$$

Since this result is time-independent, it is valid at any time, and each set $\psi_s^{\pm}$, with $s = 1, 2$ or $s = 3, 4$, is a complete set of linearly independent solutions of the Dirac equation. Note that these complete systems are not identical, and orthonormality conditions holds for each set separately. In principle, we need only one of these sets in order to expand the field operator Ψ in terms of single-particle solutions. In order to ensure that with our initial conditions the Dirac current vanishes at $t = 0$ it is advantageous to use *both* sets in our calculations, as will be shown in the next section.

Now we can construct the quantized spinor field operator in the form

$$\begin{aligned}\Psi(x) &= \int [d\mathbf{k}] \sum_{s=1,2} [b_s(\mathbf{k})\psi^+_{\mathbf{k}s} + d_s^\dagger(-\mathbf{k})\psi^-_{\mathbf{k}s}] \\ &= \int [d\mathbf{k}] \sum_{s=3,4} [b_s(\mathbf{k})\psi^+_{\mathbf{k}s} + d_s^\dagger(-\mathbf{k})\psi^-_{\mathbf{k}s}],\end{aligned} \tag{2.20}$$

where the two lines show the field expressed in terms of the two alternative bases. Here $[d\mathbf{k}] = d^3\mathbf{k}/(2\pi)^3$. The fermion fields obey canonical anticommutation relations: $\{\Psi_\alpha(t,\mathbf{x}), \Psi_\beta^\dagger(t,\mathbf{y})\} = \delta^3(\mathbf{x}-\mathbf{y})\delta_{\alpha\beta}$. The creation and annihilation operators of each set ($r, s = 1, 2$ or $r, s = 3, 4$) will obey the standard anticommutation relations,

$$\{b_r(\mathbf{k}), b_s^\dagger(\mathbf{q})\} = \{d_r(\mathbf{k}), d_s^\dagger(\mathbf{q})\} = (2\pi)^3\delta^3(\mathbf{k}-\mathbf{q})\delta_{rs}, \tag{2.21}$$

if we impose the normalization condition

$$\psi_r^{\pm\dagger}\psi_s^{\pm} = \delta_{rs}, \tag{2.22}$$

which fixes the normalization of the mode functions.

We turn now to the calculation of the expectation value of the electric current. For the sake of simplicity we choose the initial state to be the vacuum. Using the anticommutation relations (2.21) we find

$$\begin{aligned}\langle 0|j^3|0\rangle &= \frac{e}{2}\langle 0|[\bar{\Psi}, \gamma^3\Psi]|0\rangle \\ &= \frac{e}{2}\int [d\mathbf{k}] \sum_{s=1,2} \left\{-\psi^{+\ \dagger}_{\mathbf{k}s}\gamma^0\gamma^3\psi^+_{\mathbf{k}s} + \psi^{-\ \dagger}_{\mathbf{k}s}\gamma^0\gamma^3\psi^-_{\mathbf{k}s}\right\}\ .\end{aligned} \tag{2.23}$$

Alternatively,

$$\langle 0|j^3|0\rangle = \frac{e}{2}\int[d\mathbf{k}]\sum_{s=3,4}\left\{-\psi^{+}_{\mathbf{k}s}{}^{\dagger}\gamma^0\gamma^3\psi^{+}_{\mathbf{k}s} + \psi^{-}_{\mathbf{k}s}{}^{\dagger}\gamma^0\gamma^3\psi^{-}_{\mathbf{k}s}\right\} \tag{2.24}$$

Averaging the two expressions,

$$\langle 0|j^3|0\rangle = \frac{e}{4}\int[d\mathbf{k}]\sum_{s=1}^{4}\left\{-\psi^{+}_{\mathbf{k}s}{}^{\dagger}\gamma^0\gamma^3\psi^{+}_{\mathbf{k}s} + \psi^{-}_{\mathbf{k}s}{}^{\dagger}\gamma^0\gamma^3\psi^{-}_{\mathbf{k}s}\right\}. \tag{2.25}$$

This form will be useful when we turn to the adiabatic regularization in the next section. The other components of the current are zero since the electric field is in the z direction. Using (2.14) we find

$$\gamma^0\gamma^3\psi^{\pm}_{\mathbf{k}s} = \lambda_s\left[-i\gamma^0\partial_0 + \gamma^{\perp}\mathbf{k}_{\perp} - (k_3 - eA_3)\gamma^3 + m\right]\phi^{\pm}_{\mathbf{k}s} \quad , \tag{2.26}$$

and thus (2.17) and (2.22) give

$$\begin{aligned}
& \psi^{\pm}_{\mathbf{k}s}{}^{\dagger}\gamma^0\gamma^3\psi^{\pm}_{\mathbf{k}s} = \chi^{\dagger}_s f^{*\pm}_{\mathbf{k}s}\hat{D}^{\dagger}\gamma^0\gamma^3\hat{D}f^{\pm}_{\mathbf{k}s}\chi_s \\
= \; & 2\lambda_s\left\{(k_{\perp}^2 + m^2 - {p_3}^2)f^{*\pm}_{\mathbf{k}s}f^{\pm}_{\mathbf{k}s} - \dot{f}^{*\pm}_{\mathbf{k}s}\dot{f}^{\pm}_{\mathbf{k}s} + i\lambda_s p_3(f^{*\pm}_{\mathbf{k}s}\dot{f}^{\pm}_{\mathbf{k}s} - \dot{f}^{*\pm}_{\mathbf{k}s}f^{\pm}_{\mathbf{k}s})\right\} \\
= \; & \lambda_s\left[4(k_{\perp}^2 + m^2)|f^{\pm}_{\mathbf{k}s}|^2 - 1\right],
\end{aligned} \tag{2.27}$$

where the index α is not summed over. Inserting (2.27) into (2.24) gives

$$\langle 0|j^3|0\rangle = e\sum_{s=1}^{4}\int[d\mathbf{k}](k_{\perp}^2 + m^2)\lambda_s\left(|f^{-}_{\mathbf{k}s}|^2 - |f^{+}_{\mathbf{k}s}|^2\right). \tag{2.28}$$

From (2.17) and (2.22) it can be shown [22] that

$$2(k_{\perp}^2 + m^2)\left(|f^{+}_{\mathbf{k}s}|^2 + |f^{-}_{\mathbf{k}s}|^2\right) = 1. \tag{2.29}$$

Eq. (2.29) then gives the current as

$$\langle 0|j^3|0\rangle = -2e\sum_{s=1}^{4}\int[d\mathbf{k}](k_{\perp}^2 + m^2)\lambda_s|f^{+}_{\mathbf{k}s}|^2. \tag{2.30}$$

3 Adiabatic regularization

The difficulty in solving the coupled semiclassical equations (2.3) and (2.4) originates in the fact that the expectation value of the current (2.30) diverges in the interacting theory. This infinity is related to charge renormalization. The adiabatic regularization approach is suitable for identifying and removing the ultraviolet divergences [19]. In order to isolate the ultraviolet behavior of the current integrand by adiabatic expansion, we need to express the mode equations (2.12) in a WKB-like form. The generic problem is to find a suitable parameterization of the solution of the differential equation $\ddot{u}(t) + \epsilon(t)u(t) = 0$, where in the present case ϵ is the complex quantity in square brackets in (2.12). Such a parameterization was found in [23], namely,

$$f^{+}_{\mathbf{k}s}(t) = N_{\mathbf{k}s}\frac{1}{\sqrt{2\Omega_{\mathbf{k}s}}}\exp\left\{\int_0^t\left(-i\Omega_{\mathbf{k}s}(t') - \lambda_s\frac{e\dot{A}(t')}{2\Omega_{\mathbf{k}s}(t')}\right)dt'\right\}, \qquad (3.1)$$

where $N_{\mathbf{k}s}$ are normalization constants and $\Omega_{\mathbf{k}s}$ is a real generalized frequency. (The second solution $f^{-}_{\mathbf{k}s}$ for the mode equation can be found by using its Wronskian. When choosing the form (3.1) for $f^{+}_{\mathbf{k}s}$, the second solution does not have a simple form, and for this reason we expressed the current (2.30) in terms of the positive-frequency solutions only.) By substituting (3.1) into (2.12) we obtain the WKB-like equation for $\Omega_{\mathbf{k}s}$,

$$\Omega^2_{\mathbf{k}s}(t) = -\frac{\ddot{\Omega}_{\mathbf{k}s}}{2\Omega_{\mathbf{k}s}} + \frac{3}{4}\frac{\dot{\Omega}^2_{\mathbf{k}s}}{\Omega^2_{\mathbf{k}s}} + \omega^2_{\mathbf{k}} + \left(\frac{e\dot{A}}{2\Omega_{\mathbf{k}s}}\right)^2 - \lambda_s\frac{e\ddot{A}}{2\Omega_{\mathbf{k}s}} + \lambda_s\frac{e\dot{A}\dot{\Omega}_{\mathbf{k}s}}{\Omega^2_{\mathbf{k}s}}. \qquad (3.2)$$

As in the bosonic case, the equation for Ω is a second order non-linear differential equation.

This equation enables us to study the large-momentum behavior of the solutions. It will be shown that an adiabatic expansion of (3.2) to second order is needed to identify and isolate the divergences in the current (2.30). The adiabatic expansion, an expansion in powers of $1/\omega_{\mathbf{k}}$ for large $\mathbf{k}$, is obtained by successive iteration: Inserting the zeroth order solution $\Omega^{(0)}_{\mathbf{k}s} = \omega_{\mathbf{k}}$ into the right-hand side of (3.2) one obtains Ω^2 up to second order; inserting this value to the right-hand side the fourth order is obtained, and so forth. It is not difficult to see that higher-order adiabatic approximations contain terms of higher order in $1/\omega_{\mathbf{k}}$.

Noting that

$$\ddot{\omega}_{\mathbf{k}} = \frac{-e\ddot{A}p_3}{\omega_{\mathbf{k}}} + O(1/\omega_{\mathbf{k}}), \tag{3.3}$$

we have up to second order

$$\Omega_{\mathbf{k}s} = \omega_{\mathbf{k}} - e\ddot{A}\left(\lambda_s\omega_{\mathbf{k}} - p_3\right)/4\omega_{\mathbf{k}}^3 + O(1/\omega_{\mathbf{k}}^3), \tag{3.4}$$

We shall refer to

$$\Omega_{\mathbf{k}s}^{(2),min} \equiv \left[\omega_{\mathbf{k}} - e\ddot{A}\left(\lambda_s\omega_{\mathbf{k}} - p_3\right)/4\omega_{\mathbf{k}}^3\right] \tag{3.5}$$

as the minimal second order adiabatic approximation. Using the ansatz (3.1) the current reads

$$\langle 0|j^3|0\rangle = -2e\sum_{s=1}^{4}\int[d\mathbf{k}](k_\perp^2+m^2)\lambda_s\left[\frac{|N_{\mathbf{k}s}|^2}{2\Omega_{\mathbf{k}s}}\exp\left\{-\lambda_s\int\frac{e\dot{A}(t')}{\Omega_{\mathbf{k}s}(t')}dt'\right\}\right] \tag{3.6}$$

Eqs. (2.17) and (2.22) determine the normalization constants, and we define

$$\begin{aligned}
\Gamma_s(\mathbf{k}) &\equiv \frac{|N_{\mathbf{k}s}|^2}{\Omega_{\mathbf{k}s}}\exp\left\{-\lambda_s\int\frac{e\dot{A}(t')}{\Omega_{\mathbf{k}s}(t')}dt'\right\} \\
&= \left[\omega_{\mathbf{k}}^2 + \left(\frac{-\lambda_s e\dot{A}}{2\Omega_{\mathbf{k}s}} - \frac{\dot{\Omega}_{\mathbf{k}s}}{2\Omega_{\mathbf{k}s}}\right)^2 + \Omega_{\mathbf{k}s}^2 - \lambda_s 2p_3\Omega_{\mathbf{k}s}\right]^{-1}
\end{aligned} \tag{3.7}$$

to represent the constants of the square brackets in (3.6). With the identity

$$k_\perp^2 + m^2 = [\omega_{\mathbf{k}} + p_3][\omega_{\mathbf{k}} - p_3] \tag{3.8}$$

we obtain

$$\begin{aligned}
\ddot{A} &= \langle 0|j^3|0\rangle \\
&= e\int[d\mathbf{k}]\left\{\frac{\omega_{\mathbf{k}} - p_3}{\omega_{\mathbf{k}} + e\ddot{A}(\omega_{\mathbf{k}} + p_3)/4\omega_{\mathbf{k}}^3} - \frac{\omega_{\mathbf{k}} + p_3}{\omega_{\mathbf{k}} - e\ddot{A}(\omega_{\mathbf{k}} - p_3)/4\omega_{\mathbf{k}}^3} + O(\omega_{\mathbf{k}}^{-3})\right\}.
\end{aligned} \tag{3.9}$$

At large-momentum, we approximate

$$\frac{1}{1 \pm e\ddot{A}[\omega_{\mathbf{k}} \pm p_3]/4\omega_{\mathbf{k}}^4} \simeq 1 \mp \frac{e\ddot{A}}{4\omega_{\mathbf{k}}^4}[\omega_{\mathbf{k}} \pm p_3]. \tag{3.10}$$

After we perform the angular integrations and drop terms which are odd functions of p_3, the Maxwell equation becomes

$$\begin{aligned} \ddot{A} &= \langle 0|j^3|0\rangle = -e^2\ddot{A}\int[d\mathbf{k}]\left(\frac{1}{2\omega_{\mathbf{k}}^3}-\frac{p_3^2}{2\omega_{\mathbf{k}}^5}\right)+(\textit{finite part}) \qquad (3.11)\\ &= -e^2\ddot{A}\,\delta e^2+(\textit{finite part}) \end{aligned}$$

where

$$\delta e^2 \equiv \int[d\mathbf{k}]\left(\frac{1}{2\omega_{\mathbf{k}}^3}-\frac{p_3^2}{2\omega_{\mathbf{k}}^5}\right)=\frac{1}{4\pi^2}\int_0^\infty dk\left[\frac{k^2}{(k^2+m^2)^{\frac{3}{2}}}-\frac{k^4}{3(k^2+m^2)^{\frac{5}{2}}}\right]. \qquad (3.12)$$

The current in (3.11) diverges logarithmically, as expected, with the same divergence as the vacuum polarization $\Pi(q^2=0)$. We define as usual

$$e_R^2 = e^2(1+e^2\delta e^2)^{-1}\equiv Ze^2, \qquad A_R = Z^{-1/2}A \quad , \qquad (3.13)$$

so that $eA = e_R A_R$. We can also write $Z=(1-e_R^2\delta e^2)$. Multiplying (3.6) by Ze/e_R we obtain

$$\ddot{A}_R - e_R^2\ddot{A}_R\delta e^2 = e_R\sum_{s=1}^{4}\int[d\mathbf{k}](k_\perp^2+m^2)(-\lambda_s)\Gamma_s(\mathbf{k}). \qquad (3.14)$$

Upon subtracting the last term on the left-hand side,(3.14) becomes

$$\ddot{A}_R = e_R\int[d\mathbf{k}]\left[(k_\perp^2+m^2)\sum_{s=1}^{4}(-\lambda_s)\Gamma_s(\mathbf{k})+e_R^2\ddot{A}_R\left(\frac{1}{2\omega_{\mathbf{k}}^3}-\frac{p_3^2}{2\omega_{\mathbf{k}}^5}\right)\right], \qquad (3.15)$$

so the expression for $\ddot{A}_R$ is finite (see Eq. (3.11)). Since this is so, and since Γ_s depends on e and A through the product $eA = e_R A_R$ only, the R subscripts will be omitted from now on.

Consider now the difference between the exact expression (3.15) and its adiabatic approximation. Examining (3.9), (3.10), and (3.14), we can write

$$(k_\perp^2+m^2)\sum_{s=1}^{4}(-\lambda_s)\Gamma_s(\mathbf{k}) = -\frac{2p_3}{\omega_{\mathbf{k}}}-\frac{e\ddot{A}}{2\omega_{\mathbf{k}}^5}(\omega_{\mathbf{k}}^2-\pi_3^2)+R_{\mathbf{k}}(t) \quad . \qquad (3.16)$$

$R_{\mathbf{k}}(t)$ is the difference between the exact current integrand and its minimal adiabatic approximation. At large momentum this minimal adiabatic approximation has to match the exact integrand up to terms that fall off as

$O(1/\omega_{\mathbf{k}}^3)$, so the remainder $R_{\mathbf{k}}(t)$ falls off faster. Substituting (3.16) into (3.15) and using (3.9)–(3.11) the finite Maxwell equation reads

$$\ddot{A} = e \int [d\mathbf{k}] R_{\mathbf{k}}(t). \tag{3.17}$$

Superficially it seems that the second derivative of A appears only on the left-hand side of (3.17), but in fact the subsidiary condition (3.16) defining $R_{\mathbf{k}}$ is an intrinsic part of (3.17).

We are interested in solving an initial value problem where the initial conditions for the Maxwell equation are given by

$$\dot{A}(t=0) = -E_0, \quad A(t=0) = 0, \tag{3.18}$$

and where the initial state is the adiabatic vacuum, selected by matching the exact solutions in (3.1) to their adiabatic approximation, viz.,

$$\begin{aligned} \Omega_{\mathbf{k}s}(t=0) &= \omega_{\mathbf{k}}(t=0) \\ \dot{\Omega}_{\mathbf{k}s}(t=0) &= \dot{\omega}_{\mathbf{k}}(t=0) \ . \end{aligned} \tag{3.19}$$

Non-vacuum initial conditions may be handled in a manner analogous to the bosonic case, by adding to the current expectation value non-zero particle number densities, without changing the initial conditions (3.19). As in the bosonic case the initial conditions are not completely arbitrary, because the asymptotic form of the adiabatic expansion (3.16) in which $\ddot{A}$ and $R_{\mathbf{k}}$ appear must be consistent with the finite renormalized Maxwell equation (3.17). By substituting (3.19) into (3.7) and (3.16) we find that $\ddot{A}(0) = 0$, but

$$R_{\mathbf{k}}(0) = \frac{2p_3}{\omega_{\mathbf{k}}} \left\{ 1 - \frac{1 + e^2 E_0^2 / 4\omega_{\mathbf{k}}^4}{\left[(1 + e^2 E_0^2 / 8\omega_{\mathbf{k}}^4)^2 - (e^2 E_0^2 p_3 / 8\omega_{\mathbf{k}}^5)^2 \right]} \right\} \tag{3.20}$$

is not zero. However, the integration over $\mathbf{k}$ in (3.17) is zero by the charge conjugation symmetry $p_3 \rightarrow -p_3$. Hence we find $\ddot{A}(0) = 0$ in the initial value of the Maxwell equation, and the initial conditions (3.19) are consistent with the renormalization requirements. We emphasize that choosing $R_{\mathbf{k}}(0) = 0$ is *not* consistent with the initial conditions (3.19).

Given such a set of consistent initial conditions, one can evolve the set of back-reaction equations (3.2), (3.16), and (3.17). Even though the results we present below are limited to the simpler (1+1)-dimensional case, we do have some experience with a full (3+1)-dimensional calculation for scalar electrodynamics [24]. The following technical comments should apply to the fermion problem as well. By stepping the back-reaction equations forward in time, one arrives at $\{A, \dot{A}, \Omega_{\mathbf{k}s}, \dot{\Omega}_{\mathbf{k}s}\}$ at each time. In principle, $\ddot{A}$ can be calculated without using (3.17): One can examine the asymptotic behavior of $\Gamma_s(\mathbf{k})$ at large momentum and extract both $R_{\mathbf{k}}$ and $\ddot{A}$. For instance, one can parametrize the left-hand side of (3.16) as

$$-\frac{2p_3}{\omega_{\mathbf{k}}} - \frac{e\ddot{A}}{2\omega_{\mathbf{k}}^5}(\omega_{\mathbf{k}}^2 - \pi_3^2) + \frac{\alpha}{\omega_{\mathbf{k}}^4} + \frac{\beta}{\omega_{\mathbf{k}}^5} + \ldots \quad , \tag{3.21}$$

and directly extract the value of $\ddot{A}$. The fitted value of $\ddot{A}$ depends, however, on the number of terms used to expand the above series. In addition, at some momentum a fit to a finite series in powers of $\omega_{\mathbf{k}}^{-1}$ breaks down. Instead, we propose an iterative scheme making use of (3.17). We take $R_{\mathbf{k}}$ to be zero at an extremely large momentum as a trial value, so $\ddot{A}$ is extracted from (3.16) automatically. Using this value for $\ddot{A}$ we use (3.16) to extract $R_{\mathbf{k}}$ for each $\mathbf{k}$ up to this very large momentum. Then, substituting $R_{\mathbf{k}}$ in (3.17) we get a new and slightly different value for $\ddot{A}$. This procedure may be iterated until convergence of the sequence of $\ddot{A}$, $R_{\mathbf{k}}$ is reached. Thereafter the next time step is taken.

4 Semiclassical QED in (1+1) dimensions

For the sake of simplicity we solve semiclassical QED in (1+1) dimensions for a system that initially is taken to be in an adiabatic vacuum state, with given initial electric field and zero initial electric current. We follow the same steps as in the (3+1)-dimensional case. Let us indicate the modifications for (1+1) dimensions. The γ matrices are given by

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{4.1}$$

and γ^1 plays the role that γ^3 played in the (3+1) dimensional case. For a spatially homogeneous electric field, we choose a gauge where $A^0 = 0$ and define $A^1 = A(t)$, so that the second order Dirac equation becomes

$$\left[\Box + e^2A^2(t) + 2iA(t)\partial_1 - ie\partial_0 A(t)\gamma^0\gamma^1 + m^2\right]\phi(x) = 0. \tag{4.2}$$

The Dirac equation in two dimensions has two independent solutions. Either

$$f^+_{k1}\chi_1 \quad f^-_{k1}\chi_1 \tag{4.3}$$

or

$$f^+_{k2}\chi_2 \quad f^-_{k2}\chi_2 \tag{4.4}$$

may be taken as the basis set of independent solutions. Here $\gamma^0\gamma^1\chi_s = \lambda_s\chi_s$ with $\lambda_1 = 1$ and $\lambda_2 = -1$, and ω_k^2 is now given by

$$\omega_k^2(t) \equiv p_1^2 + m^2 \quad , \quad p_1 \equiv k^1 - eA^1(t) \quad . \tag{4.5}$$

χ_s are given by

$$\chi_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \chi_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{4.6}$$

The current expectation value is

$$\ddot{A} = \langle 0|j^1|0\rangle = 2e\int \frac{dk}{2\pi} m^2\{|f^+_{k2}|^2 - |f^+_{k1}|^2\}. \tag{4.7}$$

Notice that in two dimensions there is no spin, and thus there are half as many terms as in (2.30) after summation.

Now that we have the matter field equation (2.12) coupled to the Maxwell equation (4.7), we can solve the system numerically. However, the parameter e in these equations is not yet renormalized . This renormalization can be done in the same way as in Section 3. In two dimensional QED the charge renormalization is finite and we find that $\delta e^2 = (6\pi m^2)^{-1}$. Therefore in the renormalized Maxwell equation, $\ddot{A}$ can be isolated [in contrast to (3.15)], and we obtain

$$\ddot{A}_R = \frac{e_R}{1 - e_R^2 \delta e^2} \int \frac{dk}{2\pi} m^2 \left\{ \sum_{s=1,2} (-\lambda_s \Gamma_s(k)) + 2\frac{p_1}{\omega_k} \right\} \tag{4.8}$$

$$\Gamma_s(k) \equiv \left[\omega_k^2 + \left(\frac{-\lambda_s e_R \dot{A}_R}{2\Omega_{ks}} - \frac{\dot{\Omega}_{ks}}{2\Omega_{ks}} \right)^2 + \Omega_{ks}^2 - \lambda_s 2 p_1 \Omega_{ks} \right]^{-1} . \tag{4.9}$$

The last term in the braces in (4.8) does not contribute to the integral but was included for numerical purposes. The set of equations (3.2) and (4.8) with the initial conditions (3.18) and (3.20) defines the numerical back-reaction problem.

We show in Figures 1 and 2 the time evolution of the scaled electric field $\tilde{E} \equiv eE/m^2$ and the induced current $\tilde{j} \equiv ej/m^3$ as functions of $\tau \equiv mt$. Stability was attained for these results, as well as for those presented below, with the time step $d\tau = 5 \times 10^{-4}$ and a momentum grid with $d\tilde{k} \equiv dk/m = 0.003$, forcing very long running times. With these strong initial electric fields we find that the induced current increases rapidly and becomes saturated at a constant value for some time, then it decreases until it saturates at a higher constant value in the opposite direction, after which clear plasma oscillations are seen.

The saturation in the first period is easy to understand. In a classical kinetic picture, we have have $j = 2en\langle v \rangle$, where n is the density of particles (or antiparticles) and v is their velocity. j saturates as v is driven to the speed of light by the strong electric field; then E changes its sign and the existing particles, together with the newly created particles, are accelerated in the opposite direction until again they almost reach the speed of light. Due to the additional produced particles the absolute value of j in the second plateau is larger than in the first one.

The envelope of the electric field amplitude decreases substantially only in the first few oscillations and remains almost constant at later times, i.e., the pair production happens essentially in the first stages of the evolution. In subsequent oscillations n is larger and E is weaker, and therefore the frequency of oscillations increases. For quantitative comparisons to classical plasma oscillation it should be noted that as long as the particles are relativistic the frequency of plasma oscillations depends not only on n but also on the strength of E; the weaker the field the higher the frequency, in contrast to the nonrelativistic case where the plasma frequency does not depend on the amplitude of E.

As time progresses, there develops a highly oscillatory behavior (as a function of k) for the integrand in (4.8). However, we expect that the particle creation rate decreases for weaker electric fields, and that the particle number becomes approximately conserved at late times. Let us *define* the particle number density in phase space by expanding the exact fermionic field operator Ψ in terms of the lowest, zeroth-order mode functions, $f_k^{(0)}(t)$. Then the corresponding Fock-space creation and destruction operators, $a_k^{(0)}$ and $b_k^{(0)}$, become time-dependent, and

$$n(k;t) \equiv \langle 0|a_k^{(0)\dagger}(t)a_k^{(0)}(t)|0\rangle \tag{4.10}$$

may be computed by a Bogolyubov transformation (see Appendix A). The result is shown in Figures 3(a) and 4(a). Clearly, the occupation number does not exceed one in accordance with the Pauli principle.

In modelling the creation of a quark-gluon plasma in heavy-ion collisions, a flux-tube model has been used, employing a relativistic Boltzmann-Vlasov equation with a particle source term. As in our calculation, one studies the evolution of a system with a homogeneous (color) electric field, which initially contains no particles [10–13] . Let us compare the results of this phenomenological model to those of our semiclassical analysis. The relativistic kinetic equation in the presence of a homogeneous electric field is

$$\frac{\partial f}{\partial t} + eE\frac{\partial f}{\partial p} = \frac{dN}{dt\,dx\,dp} = -|eE(t)|\log\left[1 - \exp\left(-\frac{\pi m^2}{|eE(t)|}\right)\right]\delta(p), \tag{4.11}$$

where $f(p,t)$ is the (x-independent) classical phase space distribution, expressed as a function of the *kinetic* momentum p, and the right-hand side is the fermion pair-production rate in (1+1) dimensions. We assume that the particles are produced at rest [20], i.e., the source term is proportional to $\delta(p)$. (The singularity on the right hand side of (4.11) for $m = 0$ is spurious in that there can be no pair production from a classical homogeneous electric field for massless fermions in one spatial dimension. This selection rule is not, however, built into the usual WKB arguments [6] used to determine tunneling rates based on energy balance alone.) Initially, $f(p,0) = 0$. Eq. (4.11) may be solved using the characteristics $\frac{dp}{dt} = eE$, giving

$$f(p,t) = -\int_0^t dt' \, |eE(t')| \, \log\left[1 - \exp\left(-\frac{\pi m^2}{|eE(t')|}\right)\right] \, \delta(p - eA(t') + eA(t)). \tag{4.12}$$

The δ function allows us to perform the integration, and hence

$$f(p,t) = -\sum_i \log\left[1 - \exp\left(-\frac{\pi m^2}{|eE(t_i)|}\right)\right] \quad , \tag{4.13}$$

where the t_i's fulfill $p + eA(t) - eA(t_i) = 0$ and $t_i < t$.

The field equation for A is

$$\frac{d^2 A}{dt^2} = j_{total} = j_{cond} + j_{pol} \tag{4.14}$$

where the conduction current is

$$j_{cond} = 2e \int \frac{dp}{2\pi} \frac{p}{\epsilon_p} f(p,t) \quad , \tag{4.15}$$

with $\epsilon_p \equiv \sqrt{p^2 + m^2}$, and the polarization current is [12]

$$j_{pol} = \frac{2}{E} \int \frac{dp}{2\pi} \epsilon_p \frac{dN}{dt\, dx\, dp} \quad . \tag{4.16}$$

(The factors of 2 in (4.15) and (4.16) account for the contributions of the antiparticles.) Inserting (4.12) into (4.15) reduces the system to a single equation,

$$\begin{aligned} \frac{d^2 \tilde{A}}{d\tau^2} \; &= \; -\frac{e^2}{\pi m^2} \int_0^\tau d\tau' \frac{\tilde{A}(\tau') - \tilde{A}(\tau)}{\sqrt{\left[\tilde{A}(\tau') - \tilde{A}(\tau)\right]^2 + 1}} \left|\tilde{E}(\tau')\right| \log\left[1 - \exp\left(-\frac{\pi}{\left|\tilde{E}(\tau')\right|}\right)\right] \\ & \quad -\frac{e^2}{\pi m^2} \operatorname{sign}\left(\tilde{E}(\tau)\right) \log\left[1 - \exp\left(-\frac{\pi}{\left|\tilde{E}(\tau)\right|}\right)\right] \end{aligned} \tag{4.17}$$

in terms of the dimensionless variables $\tilde{A} \equiv eA/m$, $\tilde{E}$, and τ.

The time evolutions of $\tilde{E}$ and $\tilde{j}$ are shown in the dashed curves of Figures 1(a) and 2(a). In the former, we see that for an initial field $\tilde{E}_{\tau=0} = 1$ there is good quantitative agreement between the results obtained with the two very different methods. The oscillations are *slower* and the electric fields decay more *slowly* in the semiclassical calculation than in the Boltzmann-Vlasov model. For $\tilde{E}_{\tau=0} = 4$ the plasma frequencies turn out to very different in the two calculations.

The kinetic theory can be improved by use of a source term that takes Pauli blocking into account. We replace the right-hand side of (4.11) with (see appendix B)

$$\frac{dN}{dt\,dx\,dp} = -\left(1 - 2f(p,t)\right)|eE(t)| \log\left[1 - \exp\left(-\frac{\pi m^2}{|eE(t)|}\right)\right] \delta(p) \quad . \tag{4.18}$$

With this source term the agreement between the kinetic theory and the quantum theory is even more striking, as demonstrated by the dashed-dotted lines in Figures 1(b) and 2(b). The improvement is especially dramatic in the strong-field case, Fig. 2.

The amplitude of the electric field approaches a limiting value after a few oscillations, meaning that thereafter the production of particles is negligible. In the boson case an analogous effect is seen when Bose enhancement [20] is considered, but it sets in somewhat later than for fermions. The constant amplitude reflects the absence of pair creation from virtual photons and the exponentially small spontaneous pair creation at this stage of the evolution. The fact that the electric field reaches its limiting value more quickly for fermions than for bosons may be due to the difficulty of producing more fermions once the low-momentum states have been occupied. Our semiclassical description is equivalent [19] to QED with a large number N of flavors. In the large N limit any processes involving virtual photons are excluded. To include pair production via a virtual photon one must go to the next order in a $1/N$ expansion. A systematic $1/N$ expansion for the pair production problem can be obtained by using Schwinger's closed time loop generating functional [25].

The distribution function $f(p,t)$, measured after the electric field has reached an almost constant amplitude, may be compared to the quantum theory's $n(k)$ after the latter is smoothed, as shown in Figures 3 and 4. The curves have a relative displacement due to the slightly different value of A. This comparison displays the necessity of including the Pauli blocking term in the phenomenological model. In the absence of this term the occupation number exceeds one if the initial electric field is strong enough as can be seen in Fig. 4(b). This is to be expected since the source term in (4.10) is very large for a strong electric field and it is the $(1-2f)$ term that suppresses the violation of the Pauli exclusion principle.

Because of the striking similarity between the results in Figures 3(c) and 4(c), one can use the kinetic theory model to explain detailed features of the particle distribution as has often been assumed [9-13] in the past. Past treatments have not, however, included the Pauli blocking term which is crucial for strong fields. A direct derivation of the Boltzmann-Vlasov equation through some chain of approximations beginning with a field formulation would be of great interest in elucidating the physical mechanisms involved here. A step in this direction using the Wigner distribution function has been taken in Ref. [26].

A Appendix A

To obtain the number of particles per unit volume of phase space we consider

$$n(\mathbf{k};t) = \sum_{s=1,2} \langle 0| b_s^{(0)\dagger}(\mathbf{k};t) b_s^{(0)}(\mathbf{k};t)|0\rangle \ , \tag{1.1}$$

where we expand the field operator $\Psi(x)$ in terms of the adiabatic order zero mode functions, viz.,

$$\Psi(x) \ = \ \int [d\mathbf{k}] \sum_{r=1,2} \left[b_r^{(0)}(\mathbf{k};t) u_{r,\mathbf{k}}(t) e^{-i\int \omega_\mathbf{k} dt} + d_r^{(0)\dagger}(-\mathbf{k};t) v_{r,-\mathbf{k}}(t) e^{i\int \omega_\mathbf{k} dt}\right] e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{1.2}$$

having defined

$$u_{r,\mathbf{k}} = \begin{pmatrix} \sqrt{\frac{(\omega_\mathbf{k}+m)}{2\omega_\mathbf{k}}}\varphi^r \\ \frac{\vec{\sigma}\cdot\vec{\pi}}{\sqrt{2\omega_\mathbf{k}(\omega_\mathbf{k}+m)}}\varphi^r \end{pmatrix}, \quad v_{r,-\mathbf{k}} = \begin{pmatrix} \frac{-\vec{\sigma}\cdot\vec{\pi}}{\sqrt{2\omega_\mathbf{k}(\omega_\mathbf{k}+m)}}\varphi^r \\ \sqrt{\frac{(\omega_\mathbf{k}+m)}{2\omega_\mathbf{k}}}\varphi^r \end{pmatrix}. \tag{1.3}$$

Here φ^r are the spinors

$$\varphi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \varphi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{1.4}$$

These $u_{r,\mathbf{k}}(t)$ and $v_{r,\mathbf{k}}(t)$ have been chosen to satisfy

$$u_{r,\mathbf{k}}{}^\dagger u_{r',\mathbf{k}} = \delta_{rr'} \quad v_{r,\mathbf{k}}{}^\dagger v_{r',\mathbf{k}} = \delta_{rr'} \quad u_{r,\mathbf{k}}{}^\dagger v_{r',\mathbf{k}} = 0. \tag{1.5}$$

They are *not* the $t\to\infty$ limits of $\psi_s^\pm$. On the other hand, the field operator has the expansion (2.20),

$$\Psi(x) \ = \ \int [d\mathbf{k}] \sum_{s=1,2} [b_s(\mathbf{k})\psi^+_{\mathbf{ks}} + d_s^\dagger(-\mathbf{k})\psi^-_{\mathbf{ks}}] \ , \tag{1.6}$$

in terms of the time-independent operators $b_s(\mathbf{k})$ and $d_s^\dagger(-\mathbf{k})$. The operators $b_r^{(0)}(\mathbf{k};t)$ and $d_r^{(0)\dagger}(\mathbf{k};t)$ are related to the operators $b_s(\mathbf{k})$ and $d_s^\dagger(-\mathbf{k})$ by the following time-dependent Bogolyubov transformation:

$$\begin{aligned} b_r^{(0)}(\mathbf{k};t) \ &= \ \sum_{s=1,2} \alpha^s_{\mathbf{k}r}(t) b_s(\mathbf{k}) + \beta^s_{\mathbf{k}r}(t) d_s{}^\dagger(-\mathbf{k}) \\ d_r^{(0)\dagger}(-\mathbf{k};t) \ &= \ \sum_{s=1,2} \beta^{*s}_{\mathbf{k}r}(t) b_s(\mathbf{k}) + \alpha^{*s}_{\mathbf{k}r}(t) d_s{}^\dagger(-\mathbf{k}). \end{aligned} \tag{1.7}$$

Since we expand the field using two different bases, (A.2) and (A.6), which are distinct even without interaction, the Bogolyubov transformation (A.7) has a matrix form. From the canonical anticommutation relations it follows that

$$\sum_{r=1,2} (|\alpha^s_{\mathbf{k}r}|^2 + |\beta^s_{\mathbf{k}r}|^2) = 1, \tag{1.8}$$

independent of time.

Substituting (A.7) into (A.2) we obtain

$$\begin{aligned}\Psi(x) &= \int [d\mathbf{k}] \sum_{s=1,2} \sum_{r=1,2} \Big[b_s(\mathbf{k}) \left(\alpha^s_{\mathbf{k}r} u_{r,\mathbf{k}} e^{-i\int \omega_{\mathbf{k}} dt} + \beta^{*s}_{\mathbf{k}r} v_{r,-\mathbf{k}} e^{i\int \omega_{\mathbf{k}} dt} \right) \\ &\quad + d^\dagger_s(-\mathbf{k}) \left(\beta^s_{\mathbf{k}r} u_{r,\mathbf{k}} e^{-i\int \omega_{\mathbf{k}} dt} + \alpha^{*s}_{\mathbf{k}r} v_{r,-\mathbf{k}} e^{i\int \omega_{\mathbf{k}} dt} \right) \Big] e^{i\mathbf{k}\cdot\mathbf{x}}. \end{aligned} \tag{1.9}$$

We then identify $\psi^+_{\mathbf{k}s}$ and $\psi^-_{\mathbf{k}s}$ as

$$\begin{aligned}\psi^+_{\mathbf{k}s} &= \sum_{r=1,2} \left(\alpha^s_{\mathbf{k}r} u_{r,\mathbf{k}} e^{-i\int \omega_{\mathbf{k}} dt} + \beta^{*s}_{\mathbf{k}r} v_{r,-\mathbf{k}} e^{i\int \omega_{\mathbf{k}} dt} \right) e^{i\mathbf{k}\cdot\mathbf{x}} \\ \psi^-_{\mathbf{k}s} &= \sum_{r=1,2} \left(\beta^s_{\mathbf{k}r} u_{r,\mathbf{k}} e^{-i\int \omega_{\mathbf{k}} dt} + \alpha^{*s}_{\mathbf{k}r} v_{r,-\mathbf{k}} e^{i\int \omega_{\mathbf{k}} dt} \right) e^{i\mathbf{k}\cdot\mathbf{x}} \end{aligned} \tag{1.10}$$

The number of particles (or antiparticles) produced per unit phase space volume at a given momentum is then given by

$$n(\mathbf{k};t) = \sum_{r=1,2} \langle 0_{in} | b^{(0)\dagger}_r(\mathbf{k};t) b^{(0)}_r(\mathbf{k};t) | 0_{in} \rangle = \sum_{s=1,2} \sum_{r=1,2} |\beta^s_{\mathbf{k}r}(t)|^2. \tag{1.11}$$

We now calculate $|\beta^s_{\mathbf{k}r}|^2$ in terms of the solutions of the quadratic Dirac equation $f^+_{\mathbf{k}s}$. Multiplying the first equation in (A.10) by ${v_{r,\mathbf{k}}}^\dagger$ we find

$$\sum_{r=1,2} |{v_{r,\mathbf{k}}}^\dagger \psi^+_{\mathbf{k}s}|^2 = \sum_{r=1,2} |\beta^s_{\mathbf{k}r}|^2. \tag{1.12}$$

Using (2.12) and (2.14), $\psi^+_{\mathbf{k}s}$ can be written as

$$\psi^+_{\mathbf{k}s} \equiv \hat{D} f^+_{\mathbf{k}s} \chi_s = (i\gamma^0 \partial_0 - \gamma^i p_i + m) f^+_{\mathbf{k}s} \chi_s. \tag{1.13}$$

We may use the Dirac representation of the γ matrices to recast $\psi^+_{\mathbf{k}s}$ explicitly,

$$\hat{D} = \begin{pmatrix} (i\partial_0 + m) & 0 & -p_3 & (-p_1 + ip_2) \\ 0 & (i\partial_0 + m) & (-p_1 - ip_2) & p_3 \\ p_3 & (p_1 - ip_2) & (-i\partial_0 + m) & 0 \\ (p_1 + ip_2) & -p_3 & 0 & (-i\partial_0 + m) \end{pmatrix}, \tag{1.14}$$

and it follows that

$$\psi^+_{\mathbf{k}1} \equiv \hat{D} f^+_{\mathbf{k}1}\chi_1 = \begin{pmatrix} i\dot{f}^+_{\mathbf{k}1} + (m-p_3)f^+_{\mathbf{k}1} \\ -(p_1+ip_2)f^+_{\mathbf{k}1} \\ -i\dot{f}^+_{\mathbf{k}1} + (m+p_3)f^+_{\mathbf{k}1} \\ (p_1+ip_2)f^+_{\mathbf{k}1} \end{pmatrix}, \tag{1.15}$$

$$\psi^+_{\mathbf{k}2} \equiv \hat{D} f^+_{\mathbf{k}2}\chi_2 = \begin{pmatrix} (p_1-ip_2)f^+_{\mathbf{k}2} \\ i\dot{f}^+_{\mathbf{k}2} + (m-p_3)f^+_{\mathbf{k}2} \\ (p_1-ip_2)f^+_{\mathbf{k}2} \\ i\dot{f}^+_{\mathbf{k}2} + (-m-p_3)f^+_{\mathbf{k}2} \end{pmatrix}. \tag{1.16}$$

Let us also write $u_{r,\mathbf{k}}$ and ${v_{r,-\mathbf{k}}}^\dagger$ in their explicit form

$$\begin{aligned} {u_{1,\mathbf{k}}}^\dagger &= (\omega_\mathbf{k}+m, \quad 0, \quad p_3, \quad p_1-ip_2)/\sqrt{2\omega_\mathbf{k}(\omega_\mathbf{k}+m)} \\ {u_{2,\mathbf{k}}}^\dagger &= (0, \quad \omega_\mathbf{k}+m, \quad p_1+ip_2, \quad -p_3)/\sqrt{2\omega_\mathbf{k}(\omega_\mathbf{k}+m)} \\ {v_{1,-\mathbf{k}}}^\dagger &= (-p_3, \quad -p_1+ip_2, \quad \omega_\mathbf{k}+m, \quad 0)/\sqrt{2\omega_\mathbf{k}(\omega_\mathbf{k}+m)} \\ {v_{2,-\mathbf{k}}}^\dagger &= (-p_1-ip_2, \quad +p_3, \quad 0, \quad \omega_\mathbf{k}+m)/\sqrt{2\omega_\mathbf{k}(\omega_\mathbf{k}+m)} \end{aligned} \tag{1.17}$$

Thus, we obtain

$$\begin{aligned} |\beta^{*1}_{\mathbf{k1}}| &= |{v_{1,-\mathbf{k}}}^\dagger\psi^+_{\mathbf{k}1}| = \left|\frac{(\omega_\mathbf{k}+m+p_3)\left[\omega_\mathbf{k} f^+_{\mathbf{k}1} - i\dot{f}^+_{\mathbf{k}1}\right]}{\sqrt{2\omega_\mathbf{k}(\omega_\mathbf{k}+m)}}\right| \\ |\beta^{*1}_{\mathbf{k2}}| &= |{v_{2,-\mathbf{k}}}^\dagger\psi^+_{\mathbf{k}1}| = \left|\frac{(p_1+ip_2)\left[\omega_\mathbf{k} f^+_{\mathbf{k}1} - i\dot{f}^+_{\mathbf{k}1}\right]}{\sqrt{2\omega_\mathbf{k}(\omega_\mathbf{k}+m)}}\right| \\ |\alpha^{1}_{\mathbf{k1}}| &= |{u_{1,\mathbf{k}}}^\dagger\psi^+_{\mathbf{k}1}| = \left|\frac{(\omega_\mathbf{k}+m-p_3)\left[\omega_\mathbf{k} f^+_{\mathbf{k}1} + i\dot{f}^+_{\mathbf{k}1}\right]}{\sqrt{2\omega_\mathbf{k}(\omega_\mathbf{k}+m)}}\right| \\ |\alpha^{1}_{\mathbf{k2}}| &= |{u_{2,\mathbf{k}}}^\dagger\psi^+_{\mathbf{k}1}| = \left|\frac{-(p_1+ip_2)\left[\omega_\mathbf{k} f^+_{\mathbf{k}1} + i\dot{f}^+_{\mathbf{k}1}\right]}{\sqrt{2\omega_\mathbf{k}(\omega_\mathbf{k}+m)}}\right| \end{aligned} \tag{1.18}$$

To see the difference between the two different bases, we observe that in the free case $f^+_s = c_s e^{-i\omega_\mathbf{k} t}$, (A.15) and (A.16) are different from $u_{1,\mathbf{k}}$ and $u_{2,\mathbf{k}}$.

Using (A.18), (2.17), and (2.22) it is straightforward to show that equation (A.8) is satisfied. Finally we find that the number of particles produced is given by

$$
\begin{aligned}
n(\mathbf{k};t) &= \sum_{\substack{s=1,2 \\ or \quad s=3,4}} \frac{(\omega_{\mathbf{k}} + \lambda_s p_3)}{\omega_{\mathbf{k}}} \left[\omega_{\mathbf{k}}^2 |f_{\mathbf{k}s}^+|^2 + |\dot{f}_{\mathbf{k}s}^+|^2 - i\omega_{\mathbf{k}}(f_{\mathbf{k}s}^{*+} \dot{f}_{\mathbf{k}s}^+ - \dot{f}_{\mathbf{k}s}^{*+} f_{\mathbf{k}s}^+)\right] \\
&= \sum_{\substack{s=1,2 \\ or \quad s=3,4}} \frac{(\omega_{\mathbf{k}} + \lambda_s p_3)}{2\omega_{\mathbf{k}}} \frac{\left[\omega_{\mathbf{k}}^2 + \left(\frac{-\lambda_s e\dot{A}}{2\Omega_{\mathbf{k}_s}} - \frac{\dot{\Omega}_{\mathbf{k}_s}}{2\Omega_{\mathbf{k}_s}}\right)^2 + \Omega_{\mathbf{k}s}^2 - 2\omega_{\mathbf{k}}\Omega_{\mathbf{k}s}\right]}{\left[\omega_{\mathbf{k}}^2 + \left(\frac{-\lambda_s e\dot{A}}{2\Omega_{\mathbf{k}_s}} - \frac{\dot{\Omega}_{\mathbf{k}_s}}{2\Omega_{\mathbf{k}_s}}\right)^2 + \Omega_{\mathbf{k}s}^2 - \lambda_s 2p_3\Omega_{\mathbf{k}s}\right]}
\end{aligned}
\tag{1.19}
$$

Averaging these two summations, we find

$$
\begin{aligned}
n(\mathbf{k};t) &= \sum_{s=1}^{4} \frac{(\omega_{\mathbf{k}} + \lambda_s p_3)}{2\omega_{\mathbf{k}}} \left[\omega_{\mathbf{k}}^2 |f_{\mathbf{k}s}^+|^2 + |\dot{f}_{\mathbf{k}s}^+|^2 - i\omega_{\mathbf{k}}(f_{\mathbf{k}s}^{*+} \dot{f}_{\mathbf{k}s}^+ - \dot{f}_{\mathbf{k}s}^{*+} f_{\mathbf{k}s}^+)\right] \\
&= \sum_{s=1}^{4} \frac{(\omega_{\mathbf{k}} + \lambda_s p_3)}{4\omega_{\mathbf{k}}} \frac{\left[\omega_{\mathbf{k}}^2 + \left(\frac{-\lambda_s e\dot{A}}{2\Omega_{\mathbf{k}_s}} - \frac{\dot{\Omega}_{\mathbf{k}_s}}{2\Omega_{\mathbf{k}_s}}\right)^2 + \Omega_{\mathbf{k}s}^2 - 2\omega_{\mathbf{k}}\Omega_{\mathbf{k}s}\right]}{\left[\omega_{\mathbf{k}}^2 + \left(\frac{-\lambda_s e\dot{A}}{2\Omega_{\mathbf{k}_s}} - \frac{\dot{\Omega}_{\mathbf{k}_s}}{2\Omega_{\mathbf{k}_s}}\right)^2 + \Omega_{\mathbf{k}s}^2 - \lambda_s 2p_3\Omega_{\mathbf{k}s}\right]}.
\end{aligned}
\tag{1.20}
$$

In (1+1) dimensions the number of produced particles per unit length at momentum $\mathbf{k}$ is obtained by replacing p_3 by p_1 in Eq. (A.20) and dividing by 2. This is the quantity which is computed from our numerical solution of the equations and plotted in Figures 3(a) and 4(a).

B Appendix B

We derive here the Pauli blocking correction factor of Eq. (4.18). The operators $b_r(\mathbf{k};t+\Delta t)$ and $d_r^\dagger(-\mathbf{k};t+\Delta t)$ are related to the operators $b_r(\mathbf{k};t)$ and $d_r^\dagger(-\mathbf{k};t)$ by the following Bogolyubov transformation:

$$
\begin{aligned}
b_r(\mathbf{k};t+\Delta t) &= \alpha_{\mathbf{k}r}(t+\Delta t)\, b_r(\mathbf{k};t) + \beta_{\mathbf{k}r}(t+\Delta t)\, d_r^\dagger(-\mathbf{k};t) \\
d_{-r}^\dagger(-\mathbf{k};t+\Delta t) &= \beta_{\mathbf{k}r}^*(t+\Delta t)\, b_r(\mathbf{k};t) + \alpha_{\mathbf{k}r}^*(t+\Delta t)\, d_r^\dagger(-\mathbf{k};t).
\end{aligned} \quad (2.1)
$$

From the time-independence of the anticommutation relations it follows that

$$
|\alpha_{\mathbf{k}r}(t+\Delta t)|^2 + |\beta_{\mathbf{k}r}(t+\Delta t)|^2 = 1. \quad (2.2)
$$

Let us denote the density of particles (antiparticles) in momentum $\mathbf{k}$ and spin state r at time t by $n_{\mathbf{k}r}^{\pm}(t)$. We emphasize that the wavenumber $\mathbf{k}$ refers to the *canonical* momentum of the produced particle(s), which remains constant in time. We shall denote by $\mathbf{p}$ the corresponding time-dependent *kinetic* momentum [see (2.13)]. Using the Bogolyubov transformation (B.1) the number of particles at time $t+\Delta t$ is given by

$$
\begin{aligned}
& n(\mathbf{k};t+\Delta t) \\
= & \sum_{r=1,2} \langle n_{\mathbf{k}1}^+(t) n_{\mathbf{k}2}^+(t) n_{\mathbf{k}1}^-(t) n_{\mathbf{k}2}^-(t) | b_r^\dagger(\mathbf{k};t+\Delta t) b_r(\mathbf{k};t+\Delta t) | n_{\mathbf{k}1}^+(t) n_{\mathbf{k}2}^+(t) n_{\mathbf{k}1}^-(t) n_{\mathbf{k}2}^-(t)\rangle \\
= & \sum_{r=1,2} \left\{ |\alpha_{\mathbf{k}r}(t+\Delta t)|^2 n_{\mathbf{k}r}^+(t) + |\beta_{\mathbf{k}r}(t+\Delta t)|^2 \left(1 - n_{\mathbf{k}r}^-(t)\right)\right\} \quad (2.3) \\
= & \sum_{r=1,2} \left[n_{\mathbf{k}r}^+(t) + |\beta_{\mathbf{k}r}(t+\Delta t)|^2 \left(1 - n_{\mathbf{k}r}^+(t) - n_{\mathbf{k}r}^-(t)\right)\right].
\end{aligned}
$$

Due to the homogeneity of the problem, at each moment the number of particles is equal to the number of antiparticles with opposite quantum number r, i.e., $n_{\mathbf{k}r}^+(t) = n_{\mathbf{k}r}^-(t)$. The probability to create particles in each of the two spin states is equal, so that $|\beta_{\mathbf{k}1}(t)|^2 = |\beta_{\mathbf{k}2}(t)|^2$. Introducing $n(\mathbf{k};t) = n_{\mathbf{k}1}^+(t) + n_{\mathbf{k}2}^+(t)$ in (B.3) we obtain

$$
n(\mathbf{k};t+\Delta t) = n(\mathbf{k};t) + 2|\beta_{\mathbf{k}}(t+\Delta t)|^2[1 - n(\mathbf{k};t)]. \quad (2.4)
$$

Finally we write the rate of pair production in (3+1) dimensions as

$$
\frac{\Delta n(\mathbf{k};t)}{\Delta t} = 2R_{\mathbf{k}}[1 - n(\mathbf{k};t)]\ , \quad (2.5)
$$

where $R_{\mathbf{k}} = |\beta_{\mathbf{k}}(t+\Delta t)|^2/\Delta t$ is the rate of pair production when $n(\mathbf{k},t) = 0$. Schwinger's result for pair-production assumed that no particles are initially present. We will assume as a reasonable ansatz that even in the presence of particles $R_{\mathbf{k}}$ is given by Schwinger's form for pair production from the vacuum. Thus we assume

$$R_{\mathbf{k}} = -|eE(t)| \log\left[1 - \exp\left(-\frac{\pi(k_\perp^2 + m^2)}{|eE(t)|}\right)\right] \delta(k_{||} - eA(t)), \qquad (2.6)$$

where the particles are assumed to be created at rest, *i. e.*, with kinetic momentum $p_{||} = k_{||} - eA(t) = 0$.

The effect of Pauli blocking is incorporated by the factor multiplying this form of $R_{\mathbf{k}}$ in (B.5). In (1+1) dimensions there is no spin, and therefore the analogous result has the form

$$n(k; t + \Delta t) = n(k; t) + |\beta_k(t + \Delta t)|^2[1 - 2n(k; t)], \qquad (2.7)$$

and we can write the rate of pair production as

$$\frac{\Delta n(k;t)}{\Delta t} = R_k[1 - 2n(k;t)], \qquad (2.8)$$

where $R_k = |\beta_k(t+\Delta t)|^2/\Delta t$. Now we can identify the canonical phase space density $n(k;t)$ with the Boltzmann density function in terms of the kinetic momentum p, viz.,

$$n(k;t) = f(p,t) = f(k - eA(t), t) \ . \qquad (2.9)$$

We conclude that in the phenomenological Boltzmann-Vlasov approach, Pauli suppression is properly taken into account by the source term given in Eq. (B.8). This is precisely what we have used in (4.18).

We note that in [20] the normalization of the source term was smaller by 2π than in (4.11) and at the same time the definition of the current was larger by 2π than in Eqs. (4.15) and (4.16). When one takes the Pauli-blocking term into account in Eq. (4.18) (or the boson enhancement factor in Eq. (17) of Ref. [20]), one must use the normalization convention of this work. The numerical results presented in [20] reflect correct normalization.

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FIGURE CAPTIONS

FIGURE 1. (a) Time evolution of the scaled electric field $\tilde{E}$ and current $\tilde{\jmath}$, with initial value $\tilde{E} = 1.0$ and coupling $e^2/m^2 = 0.1$. Solid line is semiclassical QED, and dashed line is Boltzmann-Vlasov model. (b) Same as (a), but the dashed-dotted curve is the Boltzmann-Vlasov model with the Pauli blocking correction, Eq. (4.18).

FIGURE 2. As in Fig. 1, but for initial field $\tilde{E} = 4.0$.

FIGURE 3. (a) Momentum distribution of produced pairs, for the evolution shown in Fig. 1, at time $\tau = 600$. The abscissa is the scaled kinetic momentum $\tilde{p} \equiv \tilde{k} - \tilde{A}$, with $\tilde{k} \equiv k/m$. (b) Data of (a) after binning (histogram), compared with Boltzmann-Vlasov model (curve). (c) Data of (a) after binning (histogram), compared with Boltzmann-Vlasov model with Pauli blocking (curve).

FIGURE 4. The same as Fig. 3, but for the evolution shown in Fig. 2 (i.e., initial field $\tilde{E} = 4.0$), at time $\tau = 200$.