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Particle production in the central rapidity region

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ABSTRACT

We study pair production from a strong electric field in boost-invariant coordinates as a simple model for the central rapidity region of a heavy-ion collision. We derive and solve the renormalized equations for the time evolution of the mean electric field and current of the produced particles, when the field is taken to be a function only of the fluid proper time $\tau = \sqrt{t^2 - z^2}$. We find that a relativistic transport theory with a Schwinger source term modified to take Pauli blocking (or Bose enhancement) into account gives a good description of the numerical solution to the field equations. We also compute the renormalized energy-momentum tensor of the produced particles and compare the effective pressure, energy and entropy density to that expected from hydrodynamic models of energy and momentum flow of the plasma.

1 Introduction

A popular theoretical picture of high-energy heavy-ion collisions begins with the creation of a flux tube containing a strong color electric field [1]. The field energy is converted into particles as $q\bar{q}$ pairs and gluons are created by the Schwinger tunneling mechanism [2, 3, 4]. The transition from this quantum tunneling stage to a later hydrodynamic stage has previously been described phenomenologically using a kinetic theory model in which a relativistic Boltzmann equation is coupled to a simple Schwinger source term [5, 6, 7, 8]. Such a model requires justification, as does the use of Schwinger's formula in the case of an electric field which is changing rapidly because of screening by the produced particles. Our aim in this paper is to present a completely field-theoretic treatment of the electrodynamic initial-value problem which exhibits the decay of the electric field and subsequent plasma oscillations. This approach allows direct calculation of the spectrum of produced particles from first principles and comparison of the results with more phenomenological hydrodynamic models of the plasma.

Although our approach is relevant to heavy-ion collisions at best only during the period when the produced partons can be treated as almost free, the details of hadronization are not expected to affect the average flow of energy and momentum. Hence information obtained about energy flow in the weak coupling phase where our methods apply is translated to energy and momentum eventually deposited in the detector. Using further hadronization assumptions one can relate our results to the particle spectrum of the outgoing particles.

Recently [18, 19, 20, 21] we have presented a practical renormalization scheme appropriate for initial-value and quantum back reaction problems involving the production of charged pairs of bosons or fermions by a strong electric field. In those papers the electric field is restricted to be spatially homogeneous, so that all physical quantities are functions of time alone. The method used to identify the divergences is to perform an adiabatic expansion of the equations of motion for the time-dependent mode functions. The divergence in the expectation value of the current comes from the first few terms in the adiabatic expansion and can be isolated and identified as the usual coupling-constant counter term. In this manner we were able to construct finite equations for the process of pair production from a spatially homogeneous electric field, and to consider the back reaction that this pair production has on the time evolution of the electric field.

In heavy-ion collisions one is clearly dealing with a situation that is not spatially homogeneous. However, particle production in the central rapidity region can be modeled as an inside-outside cascade which is symmetric under longitudinal boosts and thus produces a plateau in the particle rapidity distributions [9, 10, 11, 12, 13]. This boost invariance also emerges dynamically in Landau's hydrodynamical model [14] and forms an essential kinematic ingredient in the subsequent models of Cooper, Frye, and Schonberg [15] and of Bjorken [12]. The flux-tube model of Low [16] and Nussinov [17] incorporates this invariance naturally.

Hence, kinematical considerations constrain the spatial inhomogeneity in the central rapidity region to a form that again allows an adiabatic expansion in a single variable, the fluid proper time $\tau = \sqrt{t^2 - z^2}$. In Landau's hydrodynamical model one finds that after a short time τ_0 the energy-momentum tensor in a comoving frame is a function only of τ. This is the essential assumption we make that allows us to apply the methods of [18, 19, 20, 21] to the heavy-ion collision problem.

In our approach the initial conditions of the fields are specified at $\tau = \tau_0$, that is, on a hyperbola of constant proper time. The comoving energy density is a function of τ only.[1] Then the electric field must also be a function solely of τ. We can apply the adiabatic regularization method to identify and remove the divergences. The simplification introduced by the boost symmetry allows us to study the renormalization of this inhomogeneous initial value problem.

A remarkable feature of Landau's model is the appearance of a scaling solution in which $v_z = z/t$. Alternatively, requiring invariance under longitudinal boosts [12, 15] also leads naturally to a scaling solution. In models based on Landau's ideas, one also assumes that it is possible to determine the particle spectra from the hydrodynamical flow of energy-momentum by identifying particle velocities with hydrodynamical velocities. Using our methods, we can assess the validity of these assumptions. We calculate numerically the evolution of the electric field in τ and study the accompanying production of pairs. As it turns out, the Boltzmann equation, when modified to reflect quantum statistics correctly, does very well at reproducing the gross features

[1] Following the usage in hydrodynamics we shall continue to refer to the coordinates τ and $\eta = \frac{1}{2} \ln \left(\frac{t+z}{t-z} \right)$ as the fluid proper time and the fluid rapidity, respectively, or simply as comoving coordinates.

of the field-theoretic solution. This is not too surprising, since our field equations are mean field equations (formally derived in the large-N limit), and hence are semiclassical in nature. The field theoretic treatment presented here is the first term in a systematic $1/N$ expansion in the number of parton species or quark flavors. The next order in the expansion contains dynamical gauge fields as well as charged particles in the background classical field. Thus in the next order one can study equilibration due to scattering; one could also calculate, for example, lepton production and correlations in the evolving plasma from first principles. These systematic corrections require the use of the Schwinger-Keldysh formalism [22] and will be discussed elsewhere.

The outline of the paper is as follows. In Section 2 we formulate electrodynamics in the semiclassical limit in the (τ, η) coordinate system corresponding to the hydrodynamical scaling variables. This curvilinear coordinate system requires some formalism borrowed from the literature of quantum fields in curved spaces, which we review for the benefit of the reader unfamiliar with the subject. In Section 3 we perform the renormalization of the current using the adiabatic method of our previous papers. This is needed as the source term for a finite Maxwell equation of the particle back reaction on the electric field. Section 4 is devoted to the renormalization of the energy-momentum tensor of the produced pairs in the comoving frame. We also discuss there the relationship to the effective hydrodynamic point of view. The detailed results of numerical calculations in the (1+1)-dimensional case for both charged scalars and fermions are presented in Section 5.

The paper contains three appendices. In Appendix A we present the necessary formulae used in computing the particle spectrum from the time evolution of the field modes. In Appendix B we prove that, for the boost-invariant kinematics of this problem, the distribution of particles in fluid rapidity is the same as the distribution of particles in particle rapidity. In Appendix C we reformulate the problem in the conformal time coordinate which turns out to be somewhat more convenient for actual numerical methods.

2 Electrodynamics in comoving coordinates

2.1 Scalars

We consider first the electrodynamics of spin-0 bosons. We shall use the metric convention $(- + ++)$ which is commonly used in the curved-space literature. The action in general curvilinear coordinates with metric $g_{\mu\nu}$ is

$$S = \int d^4x \sqrt{-g} \left[-g^{\mu\nu} (\nabla_\mu \phi)^\dagger \nabla_\nu \phi - m^2 \phi^\dagger \phi - \frac{1}{4} g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma} \right] , \tag{2.1}$$

where

$$\nabla_\mu \phi = (\partial_\mu - ieA_\mu)\phi \, , \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \, . \tag{2.2}$$

We use Greek indices for curvilinear coordinates and Latin indices for flat Minkowski coordinates.

To express the boost invariance of the system it is useful to introduce the light-cone variables τ and η, which will be identified later with fluid proper time and rapidity . These coordinates are defined in terms of the ordinary lab-frame Minkowski time t and coordinate along the beam direction z by

$$z = \tau \sinh \eta \quad , \quad t = \tau \cosh \eta \, . \tag{2.3}$$

The Minkowski line element in these coordinates has the form

$$ds^2 = -d\tau^2 + dx^2 + dy^2 + \tau^2 d\eta^2 \, . \tag{2.4}$$

Hence the metric tensor is given by

$$g_{\mu\nu} = \mathrm{diag}(-1, 1, 1, \tau^2). \tag{2.5}$$

with its inverse determined from $g^{\mu\nu} g_{\nu\rho} = \delta^\mu_\rho$ This metric is a special case of the Kasner metric [23].

For our future use we introduce the vierbein V^a_μ which transforms the curvilinear coordinates to Minkowski coordinates,

$$g_{\mu\nu} = V^a_\mu V^b_\nu \eta_{ab} \, , \tag{2.6}$$

where $\eta_{ab} = \mathrm{diag}\{-1, 1, 1, 1\}$ is the flat Minkowski metric. A convenient choice of the vierbein for the metric (2.5) for our problem is

$$V^a_\mu = \mathrm{diag}\{1, 1, 1, \tau\} \tag{2.7}$$

so that

$$V_a^\mu = \mathrm{diag}\left\{1, 1, 1, \frac{1}{\tau}\right\} . \tag{2.8}$$

Thus the determinant of the metric tensor is given by

$$\det V = \sqrt{-g} = \tau \, . \tag{2.9}$$

The Klein-Gordon equation is

$$\frac{1}{\sqrt{-g}} \nabla_\mu (g^{\mu\nu} \sqrt{-g}) \nabla_\nu \phi - m^2 \phi = 0 \, , \tag{2.10}$$

and Maxwell's equations read

$$\frac{1}{\sqrt{-g}} \partial_\nu (\sqrt{-g} F^{\mu\nu}) = j^\mu \, , \tag{2.11}$$

where

$$j^\mu = \mathcal{C}\{-ie[\phi^\dagger \partial^\mu \phi - (\partial^\mu \phi^\dagger)\phi] - 2e^2 A^\mu (\phi^\dagger \phi)\} \, . \tag{2.12}$$

Here $\mathcal{C}$ denotes the operation of charge symmetrization as discussed in [18].

We are interested in the case where the electric field is in the z direction and is a function of τ only. In Minkowski coordinates the only nonvanishing components of the electromagnetic field tensor F_{ab} are

$$F_{zt} = -F_{tz} \equiv E(\tau) \, . \tag{2.13}$$

In the curvilinear coordinate system we have

$$F_{\eta\tau} = -\frac{dA_\eta(\tau)}{d\tau} \, , \tag{2.14}$$

where we have chosen the gauge $A_\tau = 0$ so that the only nonvanishing component of A_μ is $A_\eta(\tau) \equiv A$. Using the relationship between the two coordinate systems we find that

$$E(\tau) = \frac{F_{\eta\tau}}{\tau} = -\frac{1}{\tau} \frac{dA}{d\tau} \, . \tag{2.15}$$

In this gauge and coordinates the Klein-Gordon equation becomes

$$\left(\partial_\tau^2 + \frac{1}{\tau} \partial_\tau - \frac{1}{\tau^2} (\partial_\eta - ieA(\tau))^2 - {\partial_x}^2 - {\partial_y}^2 + m^2 \right) \phi(\tau, \eta, x, y) = 0 \, . \tag{2.16}$$

In order to remove first derivatives with respect to τ, we define a rescaled field χ by

$$\phi = \frac{1}{\sqrt{\tau}}\chi \,. \tag{2.17}$$

The Klein-Gordon equation for χ is then

$$\left(\partial_\tau^2 - \frac{1}{\tau^2}\left[(\partial_\eta - ieA(\tau))^2 - \frac{1}{4}\right] - {\partial_x}^2 - {\partial_y}^2 + m^2\right)\chi(\tau,\eta,x,y) = 0 \,. \tag{2.18}$$

We are interested in the solution of this field equation, where A is regarded as a classical field determined by the expectation value of the Maxwell equation (2.11). This approximation ignores processes with photon propagators, and can be shown [18] to be the leading order in a large-N expansion, where N is the number of flavors of the charged scalar field.

These equations are to be solved subject to initial conditions at $\tau = \tau_0$. We need to specify the initial value of the electric field and the density matrix describing the initial state of the charged scalar field. For the problem at hand it is sufficient to describe the charged scalar field by the particle-number density and pair-correlation density with respect to an adiabatic vacuum state [see (2.30) below].

In the gauge we have chosen there is homogeneity in η as well as in the directions x and y. This allows us to introduce a Fourier decomposition for the quantum field operator χ at proper time τ,

$$\chi(\tau,\eta,\mathbf{x}_\perp) = \int [d\mathbf{k}] \left[f_\mathbf{k}(\tau)a_\mathbf{k}e^{i\mathbf{k}\cdot\mathbf{x}} + f^*_{-\mathbf{k}}(\tau)b^\dagger_\mathbf{k}e^{-i\mathbf{k}\cdot\mathbf{x}}\right] \,, \tag{2.19}$$

where

$$\begin{aligned} [d\mathbf{k}] &= \frac{dk_\eta d^2\mathbf{k}_\perp}{(2\pi)^3} \,, \\ \mathbf{k}\cdot\mathbf{x} &= k_\eta\eta + \mathbf{k}_\perp\cdot\mathbf{x}_\perp \,, \\ \mathbf{k}_\perp &\equiv (k_x,k_y) \,, \quad \mathbf{x}_\perp \equiv (x,y) \,. \end{aligned} \tag{2.20}$$

The modes $f_\mathbf{k}$ satisfy the equation

$$\frac{d^2 f_\mathbf{k}}{d\tau^2} + \omega^2_\mathbf{k}(\tau) f_\mathbf{k} = 0 \,, \tag{2.21}$$

with

$$
\begin{aligned}
\omega_{\mathbf{k}}^2(\tau) &\equiv \pi_\eta^2(\tau) + \mathbf{k}_\perp^2 + m^2 + \frac{1}{4\tau^2}\,, \\
\pi_\eta(\tau) &\equiv \frac{k_\eta - eA}{\tau}\,.
\end{aligned} \tag{2.22}
$$

We quantize the matter field by imposing commutation relations at equal τ,

$$
\left[\phi(\tau,\eta,\mathbf{x}_\perp), \frac{\partial\phi^\dagger}{\partial\tau}(\tau,\eta',\mathbf{x}'_\perp)\right] = \frac{i\delta(\eta-\eta')\delta^2(\mathbf{x}_\perp - \mathbf{x}'_\perp)}{\tau}\,. \tag{2.23}
$$

Demanding that the usual commutation relations obtain for the creation and annihilation operators,

$$
[b_{\mathbf{k}}, b_{\mathbf{k}}^\dagger] = [a_{\mathbf{k}}, a_{\mathbf{k}}^\dagger] = (2\pi)^d\delta^d(\mathbf{k}-\mathbf{k}')\,, \tag{2.24}
$$

in d spatial dimensions, we find that $f_{\mathbf{k}}$ must satisfy the condition

$$
f_{\mathbf{k}}(\tau)\partial_\tau f_{\mathbf{k}}^*(\tau) - f_{\mathbf{k}}^*(\tau)\partial_\tau f_{\mathbf{k}}(\tau) = i\,. \tag{2.25}
$$

This latter condition is satisfied by a WKB-like parametrization of $f_{\mathbf{k}}$,

$$
f_{\mathbf{k}} = \frac{e^{-i\int_0^\tau \Omega_{\mathbf{k}}(\tau')d\tau'}}{(2\Omega_{\mathbf{k}}(\tau))^{1/2}} \equiv \frac{e^{-iy_{\mathbf{k}}(\tau)}}{(2\Omega_{\mathbf{k}}(\tau))^{1/2}}\,. \tag{2.26}
$$

Because of (2.21), $\Omega_{\mathbf{k}}$ must satisfy the same differential equation as appears in our previous papers [18, 19, 20, 21],

$$
\frac{1}{2}\frac{\ddot{\Omega}_{\mathbf{k}}}{\Omega_{\mathbf{k}}} - \frac{3}{4}\frac{\dot{\Omega}_{\mathbf{k}}^2}{\Omega_{\mathbf{k}}^2} + \Omega_{\mathbf{k}}^2 = \omega_{\mathbf{k}}^2. \tag{2.27}
$$

The dot denotes differentiation with respect to τ.

The only nontrivial Maxwell equation in $A_\tau = 0$ gauge is

$$
\frac{1}{\tau}\frac{d}{d\tau}\left[\frac{1}{\tau}\frac{d}{d\tau}A(\tau)\right] = \langle j^\eta\rangle\,. \tag{2.28}
$$

In terms of the charge densities N_+ and N_- and the correlation pair density F, we can write the Maxwell equation as

$$
-\tau\frac{dE}{d\tau} = e\int[d\mathbf{k}]\frac{\pi_\eta}{\Omega_{\mathbf{k}}}[1 + 2N(\mathbf{k}) + 2F(\mathbf{k})\cos(2y_{\mathbf{k}})]\,. \tag{2.29}
$$

The structure of (2.29) is similar to that of the equation found for the homogeneous problem [18, 19]. We have used the definitions

$$\begin{aligned} \langle a^\dagger_{\mathbf{k}'} a_{\mathbf{k}} \rangle &= (2\pi)^d \delta^d(\mathbf{k}-\mathbf{k}') N_+(\mathbf{k})\,, \\ \langle b^\dagger_{-\mathbf{k}'} b_{-\mathbf{k}} \rangle &= (2\pi)^d \delta^d(\mathbf{k}-\mathbf{k}') N_-(\mathbf{k})\,, \\ \langle b_{-\mathbf{k}'} a_{\mathbf{k}} \rangle &= (2\pi)^d \delta^d(\mathbf{k}-\mathbf{k}') F(\mathbf{k})\,. \end{aligned} \tag{2.30}$$

Note that we have taken $N_+(\mathbf{k}) = N_-(\mathbf{k}) = N(\mathbf{k})$ since the current component j^τ vanishes due to the Maxwell equation (Gauss's law),

$$j^\tau = e \int \frac{[d\mathbf{k}]}{\tau} \left[N_+(\mathbf{k}) - N_-(\mathbf{k}) \right] = \frac{1}{\sqrt{-g}} \partial_\eta (\sqrt{-g} F^{\eta\tau}) = 0\,. \tag{2.31}$$

2.2 Fermions

Let us now turn to the same problem in Dirac electrodynamics. The lagrangian density for QED in curvilinear coordinates (found, for example, in [23]) gives rise to the action

$$S = \int d^{d+1}x \, (\det V) \left[\frac{-i}{2} \bar\Psi \tilde\gamma^\mu \nabla_\mu \Psi + \frac{i}{2} (\nabla^\dagger_\mu \bar\Psi) \tilde\gamma^\mu \Psi - im \bar\Psi \Psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right], \tag{2.32}$$

where [24]

$$\nabla_\mu \Psi \equiv (\partial_\mu + \Gamma_\mu - ieA_\mu)\Psi \tag{2.33}$$

and the spin connection Γ_μ is given by

$$\Gamma_\mu = \frac{1}{2} \Sigma^{ab} V_{a\nu} (\partial_\mu V^\nu_b + \Gamma^\nu_{\mu\lambda} V^\lambda_b)\,, \qquad \Sigma^{ab} = \frac{1}{4} [\gamma^a, \gamma^b]\,, \tag{2.34}$$

with $\Gamma^\nu_{\mu\lambda}$ the usual Christoffel symbol. We find that in our case (see [25])

$$\begin{aligned} \Gamma_\tau &= \Gamma_x = \Gamma_y = 0 \\ \Gamma_\eta &= -\frac{1}{2} \gamma^0 \gamma^3. \end{aligned} \tag{2.35}$$

The coordinate dependent gamma matrices $\tilde\gamma^\mu$ are obtained from the usual Dirac gamma matrices γ^a via

$$\tilde\gamma^\mu = \gamma^a V^\mu_a(x)\;. \tag{2.36}$$

The coordinate independent Dirac matrices satisfy

$$\{\gamma^\alpha, \gamma^\beta\} = 2\eta^{\alpha\beta}. \tag{2.37}$$

From the action (2.32) we obtain the Heisenberg field equation for the fermions,

$$(\tilde{\gamma}^\mu \nabla_\mu + m)\,\Psi = 0\,, \tag{2.38}$$

which takes the form

$$\left[\gamma^0\left(\partial_\tau + \frac{1}{2\tau}\right) + \gamma_\perp \cdot \partial_\perp + \frac{\gamma^3}{\tau}(\partial_\eta - ieA_\eta) + m\right]\Psi = 0\,, \tag{2.39}$$

Variation of S with respect to A_μ yields the semiclassical Maxwell equation

$$\frac{1}{\sqrt{-g}}\,\partial_\nu\left(\sqrt{-g}F^{\mu\nu}\right) = \langle j^\mu\rangle = -\frac{e}{2}\left\langle\left[\bar{\Psi}, \tilde{\gamma}^\mu\Psi\right]\right\rangle. \tag{2.40}$$

If the electric field is in the z direction and a function of τ only, we find that the only nontrivial Maxwell equation is

$$\frac{1}{\tau}\frac{dE(\tau)}{d\tau} = \frac{e}{2}\left\langle\left[\bar{\Psi}, \tilde{\gamma}^\eta\Psi\right]\right\rangle = \frac{e}{2\tau}\left\langle\left[\Psi^\dagger, \gamma^0\gamma^3\Psi\right]\right\rangle. \tag{2.41}$$

We expand the fermion field in terms of Fourier modes at fixed proper time τ,

$$\Psi(x) = \int [d\mathbf{k}] \sum_s [b_s(\mathbf{k})\psi^+_{\mathbf{k}s}(\tau)e^{ik\eta}e^{i\mathbf{p}\cdot\mathbf{x}} + d^\dagger_s(-\mathbf{k})\psi^-_{-\mathbf{k}s}(\tau)e^{-ik\eta}e^{-i\mathbf{p}\cdot\mathbf{x}}]. \tag{2.42}$$

The $\psi^\pm_{\mathbf{k}s}$ then obey

$$\left[\gamma^0\left(\frac{d}{d\tau} + \frac{1}{2\tau}\right) + i\gamma_\perp \cdot \mathbf{k}_\perp + i\gamma^3\pi_\eta + m\right]\psi^\pm_{\mathbf{k}s}(\tau) = 0, \tag{2.43}$$

where π_η has been defined previously in (2.22). The superscript $\pm$ refers to positive- or negative-energy solutions with respect to the adiabatic vacuum at $\tau = \tau_0$. Following [20], we square the Dirac equation by introducing

$$\psi^\pm_{\mathbf{k}s} = \left[-\gamma^0\left(\frac{d}{d\tau} + \frac{1}{2\tau}\right) - i\gamma_\perp \cdot \mathbf{k}_\perp - i\gamma^3\pi_\eta + m\right]\chi_s\frac{f^\pm_{\mathbf{k}s}}{\sqrt{\tau}}\,. \tag{2.44}$$

The spinors χ_s are chosen to be eigenspinors of $\gamma^0\gamma^3$,

$$\gamma^0\gamma^3\chi_s = \lambda_s\chi_s \tag{2.45}$$

with $\lambda_s = 1$ for $s = 1, 2$ and $\lambda_s = -1$ for $s = 3, 4$. They are normalized,

$$\chi_r^\dagger\chi_s = 2\delta_{rs}\,. \tag{2.46}$$

The sets $s = 1, 2$ and $s = 3, 4$ are two different complete sets of linearly independent solutions of the Dirac equation (see [20]). Inserting (2.44) into the Dirac equation (2.43) we obtain the quadratic mode equation

$$\left(\frac{d^2}{d\tau^2} + \omega_{\mathbf{k}}^2 - i\lambda_s\dot{\pi}_\eta\right) f_{\mathbf{k}s}^{\pm}(\tau) = 0, \tag{2.47}$$

where now

$$\omega_{\mathbf{k}}^2 = \pi_\eta^2 + \mathbf{k}_\perp^2 + m^2. \tag{2.48}$$

If the canonical anti-commutation relations are imposed on the Fock space mode operators, then the $\psi_{\mathbf{k}s}^{\pm}$ must obey the orthonormality relations

$$\begin{aligned} \psi_{\mathbf{k}r}^{-\dagger}\psi_{\mathbf{k}s}^{+} &= \psi_{\mathbf{k}r}^{+\dagger}\psi_{\mathbf{k}s}^{-} = 0\,, \\ \psi_{\mathbf{k}r}^{+\dagger}\psi_{\mathbf{k}s}^{+} &= \psi_{\mathbf{k}r}^{-\dagger}\psi_{\mathbf{k}s}^{-} = \frac{\delta_{r,s}}{\tau}\,, \end{aligned} \tag{2.49}$$

where $r, s = 1, 2$ or $3, 4$. Using the orthonormality relations and (2.44) we find, for a given $\mathbf{k}$ and s,

$$\omega^2 f^{*\alpha} f^\beta + \dot{f}^{*\alpha}\dot{f}^\beta - i\lambda_s\pi\left(f^{*\alpha}\dot{f}^\beta - \dot{f}^{*\alpha}f^\beta\right) = \frac{1}{2}\delta^{\alpha\beta} \tag{2.50}$$

where $\alpha, \beta = \pm$ refer to the positive and negative energy solutions. Notice that from the off-diagonal relationship ($\alpha \neq \beta$) we can express the negative-energy solutions in terms of the positive-energy ones. This fact we shall use repeatedly in what follows.

Now we parametrize the positive-energy solutions $f_{\mathbf{k}s}^{+}$ in the same manner as in Eq. (3.1) of Ref. [20],

$$f_{\mathbf{k}s}^{+}(\tau) = N_{\mathbf{k}s}\frac{1}{\sqrt{2\Omega_{\mathbf{k}s}}}\exp\left\{\int_0^\tau\left(-i\Omega_{\mathbf{k}s}(\tau') - \lambda_s\frac{\dot{\pi}_\eta(\tau')}{2\Omega_{\mathbf{k}s}(\tau')}\right)d\tau'\right\}, \tag{2.51}$$

where $\Omega_{\mathbf{k}s}$ obeys the real equation

$$\frac{1}{2}\frac{\ddot{\Omega}_{\mathbf{k}s}}{\Omega_{\mathbf{k}s}} - \frac{3}{4}\frac{\dot{\Omega}^2_{\mathbf{k}s}}{\Omega^2_{\mathbf{k}s}} + \frac{\lambda_s}{2}\frac{\ddot{\pi}_\eta}{\Omega_{\mathbf{k}s}} - \frac{1}{4}\frac{\dot{\pi}^2_\eta}{\Omega^2_{\mathbf{k}s}} - \lambda_s\frac{\dot{\pi}_\eta\dot{\Omega}_{\mathbf{k}s}}{\Omega^2_{\mathbf{k}s}} = \omega^2_{\mathbf{k}}(\tau) - \Omega^2_{\mathbf{k}s}\,. \qquad (2.52)$$

Returning to the Maxwell equation and following steps (2.27)–(2.30) of Ref. [20] we obtain

$$\frac{1}{\tau}\frac{dE(\tau)}{d\tau} = -\frac{2e}{\tau^2}\sum_{s=1}^{4}\int [d\mathbf{k}](\mathbf{k}^2_\perp + m^2)\lambda_s|f^+_{\mathbf{k}s}|^2, \qquad (2.53)$$

where we have taken the particle number $N(\mathbf{k}s)$ and pair correlation density $F(\mathbf{k}s)$ defined by the analogs of eqs. (2.30) equal to zero for simplicity.

Using the normalization conditions (2.49)–(2.50) we may express the mode functions and current in terms of the generalized frequency functions $\Omega_{\mathbf{k}s}$,

$$2|f^+_{\mathbf{k}s}|^2 = \left[\omega^2_{\mathbf{k}} + \Omega^2_{\mathbf{k}s} + \left(\frac{\dot{\Omega}_{\mathbf{k}s} + \lambda_s\dot{\pi}_\eta}{2\Omega_{\mathbf{k}s}}\right)^2 + 2\lambda_s\pi_\eta\Omega_{\mathbf{k}s}\right]^{-1}. \qquad (2.54)$$

[See Eq. (3.7) of Ref. [20].]

3 Renormalization

3.1 Scalars

The renormalization of the equations of the last section is straightforward, and is accomplished by analyzing the divergences in an adiabatic expansion of the differential equation for $\Omega_{\mathbf{k}}(\tau)$. We first present the regularization for the scalar case, where $\Omega_{\mathbf{k}}$ satisfies the differential equation (2.27). The divergences of physical quantities such as the current and the energy-momentum tensor can be isolated by expanding Ω in an adiabatic expansion. Up to second order this is given by

$$\frac{1}{\Omega} = \frac{1}{\omega} + \left(\frac{\ddot{\omega}}{4\omega^4} - \frac{3\dot{\omega}^2}{8\omega^5}\right) + \cdots \quad . \qquad (3.1)$$

The unrenormalized Maxwell equation is

$$-\tau\frac{dE}{d\tau} = e\int[d\mathbf{k}]\,\frac{\pi_\eta}{\Omega_{\mathbf{k}}(\tau)}\,[1+2N(\mathbf{k})+2F(\mathbf{k})\cos(2y_{\mathbf{k}}(\tau))]\,. \tag{3.2}$$

To study its renormalization in $d=3$ spatial dimensions we need to consider only the vacuum term,

$$-\tau\frac{dE}{d\tau} = e\int[d\mathbf{k}]\,\frac{\pi_\eta}{\Omega_{\mathbf{k}}(\tau)}\,. \tag{3.3}$$

The adiabatic expansion (3.1) leads to

$$-\tau\frac{dE}{d\tau} = e\int[d\mathbf{k}]\,\frac{(k_\eta - eA_\eta)}{\tau}\,\left[\frac{1}{\omega_{\mathbf{k}}} + \left(\frac{\ddot{\omega}_{\mathbf{k}}}{4\omega_{\mathbf{k}}^4} - \frac{3\dot{\omega}_{\mathbf{k}}^2}{8\omega_{\mathbf{k}}^5}\right)\right] + \cdots \quad . \tag{3.4}$$

The first term in (3.4) is zero by reflection symmetry if we choose fixed integration boundaries for the kinetic momentum $k_\eta - eA_\eta$. The only divergent terms occur at second order. (Higher terms in the expansion have more powers of $\mathbf{k}$ in the denominator.) Using (2.22) and reflection symmetry, the right-hand side of (3.4) can be written as

$$e\int[d\mathbf{k}]\,(k_\eta - eA)^2\left\{\frac{e\dot{A}}{\tau^4\omega_{\mathbf{k}}^5} - \frac{e\ddot{A}}{4\tau^3\omega_{\mathbf{k}}^5} - \frac{5e\dot{A}\left[(k_\eta - eA)^2 + \frac{1}{4}\right]}{4\tau^6\omega_{\mathbf{k}}^7}\right\}. \tag{3.5}$$

Performing the k_η and azimuthal angular integrations first we obtain

$$-\frac{e^2}{48\pi^2}\int_0^{\Lambda^2} dk_\perp^2\left\{\frac{\ddot{A} - \frac{\dot{A}}{\tau}}{(k_\perp^2 + m^2 + \frac{1}{4\tau^2})} + \frac{\dot{A}}{2\tau^3(k_\perp^2 + m^2 + \frac{1}{4\tau^2})^2}\right\}, \tag{3.6}$$

where we have inserted a cutoff in the remaining transverse momentum integration. The logarithmically divergent first term in (3.6) is

$$\frac{1}{24\pi^2}\left(-\ddot{A} + \frac{\dot{A}}{\tau}\right)\left[\ln\left(\frac{\Lambda}{m}\right) - \ln\left(1 + \frac{1}{4m^2\tau^2}\right)\right]. \tag{3.7}$$

We recognize the cutoff dependent infinite part as

$$e^2\delta e^2\tau\frac{dE}{d\tau}\,, \tag{3.8}$$

where

$$\delta e^2 = (1/24\pi^2)\ln(\Lambda/m) \tag{3.9}$$

is the usual one-loop charge renormalization factor in scalar QED. Defining the renormalized charge via

$$e_R{}^2 = e^2(1+e^2\delta e^2)^{-1} = e^2(1-e_R{}^2\delta e^2)\,, \tag{3.10}$$

and using the Ward identity $eE = e_R E_R$, we may absorb the divergence in the current into the left side of the Maxwell equation to obtain a finite renormalized equation suitable for numerical integration.

In the $d = 1$ case there is no transverse momentum integration, and the charge renormalization is finite. The finite coefficient of the combination $-\ddot{A} + \dot{A}/\tau$ is τ-dependent, and is given by

$$\left[12\pi\left(m^2 + \frac{1}{4\tau^2}\right)\right]^{-1}\,. \tag{3.11}$$

The standard result in (1+1) dimensions is $\delta e^2 = (12\pi m^2)^{-1}$, and this is what is obtained for the spatially homogeneous problem by our method as well [19]. Thus, absorbing δe^2 in the renormalization of the charge [see (3.10)] leaves us with a (finite) τ-dependent coefficient that is multiplied by the above combination, which appears now on both sides of the finite Maxwell equation, just as in the three dimensional case. This feature is peculiar to scalars in the τ coordinate. The actual numerical solution of the scalar equations was performed in the conformal time coordinate $u = \ln(m\tau)$ discussed in Appendix C.

3.2 Fermions

We turn to the renormalization problem in the spin-$\frac{1}{2}$ case. The unrenormalized Maxwell equation is

$$\frac{d}{d\tau}\left(\frac{1}{\tau}\frac{dA}{d\tau}\right) = -2e\sum_{s=1}^{4}\int [d\mathbf{k}](k_\perp^2 + m^2)\lambda_s \frac{|f^+_{\mathbf{k}s}|^2}{\tau}\,. \tag{3.12}$$

Replacing Ω and $\dot{\Omega}$ with ω and $\dot{\omega}$ on the left-hand-side of (2.52), we obtain the adiabatic expansion up to second order,

$$\Omega_s^2 = \omega^2 - \frac{1}{2\omega^2}\left[\pi\ddot{\pi} + \dot{\pi}^2\left(1-\frac{\pi^2}{\omega^2}\right)\right] + \frac{3}{4}\frac{\pi^2\dot{\pi}^2}{\omega^4} + \frac{\dot{\pi}^2}{4\omega^2} + \frac{\lambda_s\dot{\pi}^2\pi}{\omega^3} - \frac{\lambda_s\ddot{\pi}}{2\omega} + \cdots \tag{3.13}$$

Using this expansion in (2.54) allows us to express the integrand of (3.12) in the form

$$\sum_{s=1}^{4}(k_\perp^2+m^2)(-2\lambda_s))\frac{|f^+_{\mathbf{k}s}|^2}{\tau} = \frac{2\pi_\eta}{\tau\omega_\mathbf{k}} - \left(\frac{\ddot{\pi}_\eta}{2\omega_\mathbf{k}^5} - \frac{5\dot{\pi}_\eta^2\pi_\eta}{4\omega_\mathbf{k}^7}\right)\frac{(\omega_\mathbf{k}^2-\pi_\eta^2)}{\tau} - R_\mathbf{k}(\tau)\,, \tag{3.14}$$

where $R_\mathbf{k}(\tau)$ falls faster than ω^{-3} and so leads to a finite contribution to the current. Substituting (3.14) and using the definition (2.22) of π_η and its first and second derivative into (3.12) yields

$$\begin{aligned}\frac{dE}{d\tau} &= \frac{e^2}{2\tau^2}\int[d\mathbf{k}]\ \frac{k_\perp^2+m^2}{\omega_\mathbf{k}^5}\left\{\left(\ddot{A}-2\frac{\dot{A}}{\tau}\right)+\frac{5\dot{A}\pi_\eta^2}{\tau\omega_\mathbf{k}^2}\right\} - e\int[d\mathbf{k}]R_\mathbf{k}(\tau)\\ &= -\frac{e^2}{6\pi^2}\ln\left(\frac{\Lambda}{m}\right)\frac{dE}{d\tau} - e\int[d\mathbf{k}]R_\mathbf{k}(\tau).\end{aligned} \tag{3.15}$$

where Λ is again the cutoff in the transverse momentum integral which has been reserved for last.

The cutoff dependent term on the right-hand side is precisely the logarithmically divergent charge renormalization in $3+1$ dimensions. Defining $\delta e^2 = (1/6\pi^2)\ln(\Lambda/m)$ as usual and shifting this term to the left-hand side we obtain

$$e\frac{dE}{d\tau}(1+e^2\delta e^2) = -e^2\int[d\mathbf{k}]R_\mathbf{k}(\tau)\,, \tag{3.16}$$

after multiplying both sides of the equation by e. The renormalized charge is

$$e_R^2 = \frac{e^2}{(1+e^2\delta e^2)} = Ze^2. \tag{3.17}$$

and the Ward identity assures us that $e_R E_R = eE$. Hence we obtain the renormalized Maxwell equation

$$\frac{dE_R}{d\tau} = -e_R\int[d\mathbf{k}]R_\mathbf{k}(\tau), \tag{3.18}$$

where $R_\mathbf{k}(\tau)$ is defined by Eq. (3.14), and the integral is now completely convergent.

4 The Energy-Momentum Tensor and Effective Hydrodynamics

In our semiclassical calculations, we follow the evolution of the matter and electromagnetic fields. With these quantities in hand we can calculate, in addition to the particle spectrum, other physical quantities such as the energy density and the longitudinal and transverse pressures. To obtain these quantities we derive the energy-momentum tensor in the comoving frame. We shall give explicit formulae here only for the fermionic case.

The energy-momentum tensor for QED is obtained by varying the action in (2.32). We find

$$T_{\mu\nu}^{total} = \frac{-2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} = T_{\mu\nu}^{fermion} + T_{\mu\nu}^{em} \tag{4.1}$$

with

$$\begin{aligned} T_{\mu\nu}^{fermion} &= \frac{i}{4}\left[\bar{\Psi}, \tilde{\gamma}_{(\mu}\nabla_{\nu)}\Psi\right] - \frac{i}{4}\left[\nabla_{(\mu}\bar{\Psi}, \tilde{\gamma}_{\nu)}\Psi\right] \\ T_{\mu\nu}^{em} &= -\left(\frac{1}{4}g_{\mu\nu}F^{\rho\sigma}F_{\rho\sigma} + F_{\mu}{}^{\rho}F_{\rho\nu}\right). \end{aligned} \tag{4.2}$$

In the following we shall drop the superscript on the fermion part of the energy-momentum tensor where it causes no confusion to do so.

We are interested in calculating the diagonal terms of the matter part of the energy-momentum tensor and in identifying them with the energy and pressure in the comoving frame. We begin with $T_{\tau\tau}$,

$$\begin{aligned} \langle 0|T_{\tau\tau}|0\rangle &= \frac{i}{\tau}\sum_{s=1}^{2}\int [d\mathbf{k}]\,(k_\perp^2+m^2)\left[f_{\mathbf{k}s}^{*-}\,\frac{\overleftrightarrow{d}}{d\tau}\,f_{\mathbf{k}s}^{-} - f_{\mathbf{k}s}^{*+}\,\frac{\overleftrightarrow{d}}{d\tau}\,f_{\mathbf{k}s}^{+}\right] \\ &= -\frac{i}{\tau}\sum_{s=1}^{2}\int [d\mathbf{k}]\,\left[2(k_\perp^2+m^2)f_{\mathbf{k}s}^{*+}\,\frac{\overleftrightarrow{d}}{d\tau}\,f_{\mathbf{k}s}^{+} + \frac{\lambda_s\pi_\eta}{\tau}\right]. \end{aligned} \tag{4.3}$$

In the latter form only the positive frequency mode functions appear which is most useful for the adiabatic expansion below. Averaging over $s = 1, 2$ and $s = 3, 4$ we may also write (4.3) in the form

$$\langle 0|T_{\tau\tau}|0\rangle = -2\sum_{s=1}^{4}\int [d\mathbf{k}](k_\perp^2+m^2)\frac{\Omega_{\mathbf{k}s}}{\tau}|f_{\mathbf{k}s}^{+}|^2. \tag{4.4}$$

This expression contains quartic and quadratic divergences in $3+1$ dimensions present even in the complete absence of fields (the vacuum energy or cosmological constant terms) and a logarithmic divergence which is related to the charge renormalization of the last section. To isolate these divergent terms we express the integrand of $T_{\tau\tau}$ as the sum of its second order adiabatic expansion and a remainder term,

$$-2\sum_{s=1}^{4}(k_\perp^2+m^2)\frac{\Omega_{\mathbf{k}s}}{\tau}|f_{\mathbf{k}s}^{+}|^2 = -\frac{2\omega}{\tau} + (k_\perp^2+m^2)\frac{(\pi_\eta+e\dot{A})^2}{4\omega^5\tau^3} + R_{\tau\tau}(\mathbf{k}), \quad (4.5)$$

where $R_{\tau\tau}(\mathbf{k})$ falls off faster than ω^{-3} so that the integral over $R_{\tau\tau}(\mathbf{k})$ is finite.

The first term in (4.5) gives rise to a quartic divergence in 3 space dimensions (or a quadratic divergence in 1 space dimension) independent of the electric field and must be subtracted. The π_η^2 term in (4.5) gives rise to a quadratic divergence in 3 dimensions which must be likewise subtracted. However, in 1 space dimension this term yields a *finite* contribution to $\langle 0|T_{\tau\tau}|0\rangle$ which must be retained, since it is τ dependent, and cannot be absorbed into a cosmological constant counterterm. Strictly speaking, subtracting this term in 3 dimensions can only be justified by using a coordinate invariant regularization scheme for formally divergent quantities, such as dimensional regularization, where quartic and quadratic divergences are automatically excised. Only in such a scheme can one be certain that the divergence can be absorbed into a counterterm of the generally coordinate invariant lagrangian in (2.32). The net result is that this term must be handled somewhat differently in the 3 and 1 dimensional cases.

The term which is linear in $\dot{A}$ vanishes when integrated symmetrically. The term in (4.5) proportional to $e^2\dot{A}^2$ is logarithmically divergent in 3 dimensions, and finite in 1 dimension. In fact it is precisely

$$\delta e^2\frac{\dot{A}^2}{2\tau^2}\,, \quad (4.6)$$

where δe^2 is given by (3.9) in 3 dimensions and by $(12\pi m^2)^{-1}$ in 1 dimension. In either case, it can be absorbed into a renormalization of the electric energy term of the stress tensor,

$$T_{\tau\tau}^{em} = \frac{\dot{A}^2}{2\tau^2}\ . \quad (4.7)$$

by charge renormalization. When added to the electromagnetic term it gives

$$(1 + e^2 \delta e^2) \frac{\dot{A}^2}{2\tau^2} = Z^{-1} \frac{\dot{A}^2}{2\tau^2} = \frac{E_R^2}{2}. \tag{4.8}$$

Thus the explicitly finite, renormalized $\langle T_{\tau\tau} \rangle$ for the combined matter and electromagnetic system is

$$\langle T_{\tau\tau} \rangle = \frac{E_R^2}{2} + \int [d\mathbf{k}] R_{\tau\tau}(\mathbf{k}) \,, \tag{4.9}$$

in three dimensions where $R_{\tau\tau}(\mathbf{k})$ is defined in (4.5). In one dimension the finite π_η^2 term that must be retained in the adiabatic expansion of (4.5) gives rise to an additional $(12\pi\tau^2)^{-1}$ on the right side of (4.9).

Turning now to $T_{\eta\eta}$, the matter part is given by

$$\langle 0 | T_{\eta\eta} | 0 \rangle = 2\tau \sum_{s=1}^{4} \int [d\mathbf{k}] (k_\perp^2 + m^2) \lambda_s \pi_\eta |f_{\mathbf{k}s}^+|^2. \tag{4.10}$$

The adiabatic expansion of the integrand gives in this case

$$\begin{aligned} 2\tau \sum_{s=1}^{4} (k_\perp^2 + m^2) \lambda_s \pi_\eta |f_{\mathbf{k}s}^+|^2 &= -\frac{2\pi_\eta^2 \tau}{\omega_\mathbf{k}} + \left[\frac{\ddot{\pi}_\eta}{2\omega_\mathbf{k}^5} - \frac{5\dot{\pi}_\eta^2 \pi_\eta}{4\omega_\mathbf{k}^7} \right] \pi_\eta \tau (\omega_\mathbf{k}^2 - \pi_\eta^2) \\ &\quad + \pi_\eta \tau^2 R_\mathbf{k}(\tau) \,. \end{aligned} \tag{4.11}$$

Inserting this into (4.10) we find again that the quadratic and quartic divergences in 3 dimensions are independent of the electric field and that the terms proportional to $\dot{A}$ vanish, whereas the term proportional to $\dot{A}^2$ again renormalizes the electromagnetic contribution to the energy-momentum tensor,

$$T_{\eta\eta}^{em} = -\frac{\dot{A}^2}{2} \,. \tag{4.12}$$

Therefore the fully renormalized total $T_{\eta\eta}$ in 3 dimensions is

$$\langle T_{\eta\eta} \rangle = -\frac{1}{2} E_R^2 \tau^2 + \tau^2 \int [d\mathbf{k}] \, \pi_\eta R_\mathbf{k}(\tau) \,. \tag{4.13}$$

As for $\langle T_{\tau\tau} \rangle$ there is a finite additional term in 1 dimension that must be added to this expression, which in this case is equal to a constant, $(12\pi)^{-1}$.

For the transverse components of the matter stress tensor in 3 dimensions we have

$$\begin{aligned}\langle T_{xx}\rangle &= \langle T_{yy}\rangle \\ &= -\frac{1}{2\tau}\sum_{s=1}^{4}\int [d\mathbf{k}] k_\perp^2 \left[\left(i f^{*+}_{\mathbf{k}s} \overleftrightarrow{\frac{d}{d\tau}} f^{+}_{\mathbf{k}s}\right) + \lambda_s \pi_\eta |f^{+}_{\mathbf{k}s}|^2\right]. \end{aligned} \tag{4.14}$$

In a precisely analogous manner one develops the adiabatic expansion of the integrand, isolates the divergences and the logarithmically divergent charge renormalization which combines with the electromagnetic stress,

$$T^{em}_{xx} = T^{em}_{yy} = \frac{\dot{A}^2}{2\tau^2}\ . \tag{4.15}$$

and arrives at the renormalized form for the total $\langle T_{xx}\rangle = \langle T_{yy}\rangle$ can be written as

$$\langle T_{xx}\rangle = \frac{1}{2}\int [d\mathbf{k}] \frac{k_\perp^2}{(k_\perp^2 + m^2)} [R_{\tau\tau} - \pi_\eta R_{\mathbf{k}}] + \frac{1}{2}E_R^2 \tag{4.16}$$

where $R_{\tau\tau}$ and $R_{\mathbf{k}}$ have been defined previously. This result for the transverse components may be obtained by consideration of the trace of the energy momentum tensor T^μ_μ. From Eqs. (4.9), (4.13), and (4.16) we note that this trace vanishes as $m \to 0$.

Both the unrenormalized and renormalized total energy-momentum tensors are covariantly conserved, so that we have

$$\begin{aligned} T^{\mu\nu}{}_{;\mu} &= (T^{\mu\nu}{}_{;\mu})_{matter} + (T^{\mu\nu}{}_{;\mu})_{em} = 0\,, \\ (T^{\mu\nu}{}_{;\mu})_{em} &= -F^{\nu}_{\ \mu} j^\mu. \end{aligned} \tag{4.17}$$

In the boost invariant proper time coordinates one may verify that this equation takes the form,

$$\partial_\tau T_{\tau\tau} + \frac{T_{\tau\tau}}{\tau} + \frac{T_{\eta\eta}}{\tau^3} = F_{\eta\tau} j^\eta\,. \tag{4.18}$$

If we follow the standard practice and define the energy density and transverse and longitudinal pressures via,

$$T_{\mu\nu} = \mathrm{diag}(\epsilon, p_\perp, p_\perp, p_\parallel \tau^2), \tag{4.19}$$

then the energy conservation equation takes the form

$$\frac{d(\epsilon\tau)}{d\tau} + p_{\|} = E j_{\eta} \tag{4.20}$$

In most hydrodynamical models of particle production, one usually assumes that thermal equilibrium sets in and that there is an equation of state $p_{\|} = p_{\|}(\epsilon)$. For boost-invariant kinematics $v = z/t$, all the collective variables are functions only of τ and therefore $p_{\|}$ is implicitly a function of ϵ. In our field-theory model in the semiclassical limit there is no real scattering of partons and thus one does not have charged particles in equilibrium or a true equation of state. Hence the transverse and longitudinal pressures are different. In the next order in $1/N$ there is parton-parton scattering and it remains to be seen whether thermal equilibrium and isotropy of the pressures will emerge dynamically. Nevertheless, even in this order in $1/N$ one can define an effective hydrodynamics using the expectation value of the field theory's stress tensor (4.19). Formally introducing the auxiliary quantities "temperature" and "entropy density" in a suggestive way in analogy with thermodynamics via,

$$\epsilon + p_{\|} = Ts \; ; \qquad d\epsilon = T ds \tag{4.21}$$

we find that the entropy density obeys,

$$\frac{d(s\tau)}{d\tau} = \frac{E j_{\eta}}{T} \tag{4.22}$$

Notice that when the electric field goes to zero $s\tau$ becomes constant. If $p_{\|}$ also goes to zero with τ faster than $1/\tau$, then $\epsilon\tau$ is constant when the electric field goes to zero.

In standard phenomenological models of particle production such as Landau's hydrodynamical model, one usually assumes that the hydrodynamics describes an isotropic perfect fluid whose energy momentum tensor in a comoving frame has the form (4.19) with $p_{\|} = p_{\perp}$. Then one component of the energy conservation equation becomes,

$$\frac{d(s\tau)}{d\tau} = 0 \tag{4.23}$$

which is of the same form as (4.22) in the absence of electric field, with the difference that the *isotropic* pressure enters into the definitions of the entropy density and temperature of the fluid in (4.21). From these definitions one can also calculate directly the entropy density in the comoving frame as a function of τ by

$$s(\tau) = \exp\left\{\int_0^\tau \frac{1}{\epsilon + p}\frac{d\epsilon}{d\tau}\; d\tau\right\} \tag{4.24}$$

Since we follow the microscopic degrees of freedom, we can also construct the Boltzmann entropy function in terms of the single particle distribution function in comoving phase space. This is done in Appendix A.

Let us compare the energy spectra of the isotropic hydrodynamics with the results of our field theoretic approach. One quantity we wish to determine is the amount of lab frame energy in a bin of fluid rapidity, $dE_{lab}/d\eta$. For the isotropic hydrodynamics, in the lab frame one can write the energy momentum tensor in the form,

$$T^{ab} = (\epsilon + p)u^a u^b + p\eta^{ab} \tag{4.25}$$

where $u^t = \cosh\eta$ and $u^z = \sinh\eta$. Then calculating $dE_{lab}/d\eta$ on a surface of constant proper time τ we obtain,

$$\begin{aligned}
\frac{dE_{lab}}{d\eta} &= \int T^{ta}\frac{d\sigma_a}{d\eta} \\
d\sigma_a &= A_\perp(dz, -dt) = A_\perp\tau_f(\cosh\eta, -\sinh\eta) \\
\frac{dE_{lab}}{d\eta} &= A_\perp\epsilon(\tau_f)\tau_f\cosh\eta\ ,
\end{aligned} \tag{4.26}$$

where $A_\perp$ is a transverse size which in a flux tube model would be the transverse area of the chromoelectric flux tube.

We show now that our microscopic hydrodynamics gives an *identical* result, without any assumptions about thermal equilibrium. In fact, transforming the result of our field theory calculation (4.19) to the lab frame,

$$T^{ab} = \begin{pmatrix} \epsilon\cosh^2\eta + p_\parallel\sinh^2\eta & 0 & 0 & (\epsilon + p_\parallel)\cosh\eta\sinh\eta \\ 0 & p_\perp & 0 & 0 \\ 0 & 0 & p_\perp & 0 \\ (\epsilon + p_\parallel)\cosh\eta\sinh\eta & 0 & 0 & \epsilon\sinh^2\eta + p_\parallel\cosh^2\eta \end{pmatrix} \tag{4.27}$$

and recalculating $dE_{lab}/d\eta$ again gives (4.26), where $\epsilon(\tau)$ is now explicitly calculable from the modes of the field theory.

In hydrodynamic models one assumes that hadronization does not effect the collective motion. If all the particles that are produced after hadronization are pions then the number of particles in a bin of rapidity should be just the energy in a bin of rapidity divided by the energy of a single pion namely,

$$\begin{aligned} \frac{dN}{d\eta} &= \frac{1}{m_\pi \cosh\eta} \frac{dE_{lab}}{d\eta} \\ &= A_\perp \frac{\epsilon(\tau_f)\tau_f}{m_\pi} \end{aligned} \tag{4.28}$$

To see if this formula is working in our parton domain we can instead use the mass of a parton in place of m_π and check directly whether the spectrum of partons given by

$$\frac{dN}{d\eta} = \frac{\epsilon(\tau_f)\tau_f}{m} \tag{4.29}$$

agrees with explicit calculation of particle number in the field theory, as given in Appendix A.

In order for the formula (4.29) to be independent of τ_f we require that the electric field become vanishingly small and that the pressure go to zero faster than $\frac{1}{\tau}$ at large τ. Indeed we will find this is approximately true in the numerical simulations, the results of which we will present in the next section.

5 Numerical results in (1+1) dimensions

In this section we present the results of solving the back-reaction problem in two dimensions (proper time τ and fluid rapidity η), and compare the results to a phenomenological Boltzmann-Vlasov model. In previous calculations using kinetic equations in flux tube models [5, 6, 7] it was assumed that the Schwinger source term (WKB formula) can be used by taking the electric field, hitherto constant, to be a function of proper time. However, in the Schwinger derivation the time parameter (which is not boost invariant) plays an implicit role. Therefore, it is not clear *a priori* if such a source term in the kinetic equations represents the correct rate of particle production. From the experience obtained in the spatially homogeneous case, we believe that if we know the correct source term, a phenomenological Boltzmann-Vlasov approach should agree with the semiclassical QED calculation.

The phenomenological Boltzmann-Vlasov equation in $3+1$ dimensions can be written covariantly as

$$\frac{Df}{D\tau} \equiv p^\mu \frac{\partial f}{\partial q^\mu} - e p^\mu F_{\mu\nu} \frac{\partial f}{\partial p_\nu} = p_0 g^{00} \frac{dN}{dq^0 d^3\mathbf{q} d^3\mathbf{p}}\,, \tag{5.1}$$

We shall write the transport equation in the comoving coordinates and their conjugate momenta,

$$q^\mu = (\tau, x, y, \eta), \quad p_\mu = (p_\tau, p_x, p_y, p_\eta)\,. \tag{5.2}$$

In order to write the invariant source term in these coordinates, we begin with the WKB formula, which is

$$\begin{aligned} \frac{dN}{[(-g)^{1/2} dq^0 d^3\mathbf{q}] d^2\mathbf{p}_\perp} &= \pm[1 \pm 2f(\mathbf{p},\tau)] e|E(\tau)| \\ &\times \ln\left[1 \pm \exp\left(-\frac{\pi(m^2+\mathbf{p}_\perp^2)}{e|E(\tau)|}\right)\right], \end{aligned} \tag{5.3}$$

if the (constant) electric field is in the z direction. The $\pm$ refers to the cases of charged bosons or fermions respectively. Our model for the spatially homogeneous case consisted of applying this formula even for a time-dependent electric field. Here we will allow the electric field to be a function of the proper time, writing

$$\sqrt{F^{\mu\nu} F_{\mu\nu}} = |E(\tau)|\,. \tag{5.4}$$

In the spatially homogeneous case we assumed that particles are produced at rest, multiplying the WKB formula by $\delta(p_z)$. This longitudinal momentum dependence violates the Lorentz-boost symmetry. Here we assume [5, 6, 7] that the p_η distribution is $\delta(p_\eta)$, which is boost-invariant according to (B.7). Assuming boost-invariant initial conditions for f, invariance of the Boltzmann-Vlasov assures that the distribution function is a function only of the boost invariant variables $(\tau, \eta - y)$ or (τ, p_η). The kinetic equation reduces to

$$\begin{aligned} \frac{\partial f}{\partial \tau} + e F_{\eta\tau}(\tau) \frac{\partial f}{\partial p_\eta} &= \pm[1 \pm 2f(\mathbf{p},\tau)] e\tau|E(\tau)| \\ &\times \ln\left[1 \pm \exp\left(-\frac{\pi(m^2+\mathbf{p}_\perp^2)}{e|E(\tau)|}\right)\right] \delta(p_\eta). \end{aligned} \tag{5.5}$$

Turning now to the Maxwell equation, we have from (2.28) that

$$-\tau\frac{dE}{d\tau} = j_\eta = j_\eta^{cond} + j_\eta^{pol}\,, \tag{5.6}$$

where j^{cond} is the conduction current and j_μ^{pol} is the polarization current due to pair creation [8, 19, 20]. The invariant phase-space in the comoving coordinates is

$$\frac{1}{(-g)^{1/2}p_0 g^{00}}\frac{d^3\mathbf{p}}{(2\pi)^3} = \frac{d\mathbf{p}_\perp dp_\eta}{(2\pi)^3\tau p_\tau}\,. \tag{5.7}$$

Thus in (1+1) dimensions we have

$$\begin{aligned} j_\eta^{cond} &= 2e\int\frac{dp_\eta}{2\pi\tau p_\tau}p_\eta f(p_\eta,\tau) \\ j_\eta^{pol} &= \frac{2}{F^{\tau\eta}}\int\frac{dp_\eta}{2\pi\tau p_\tau}p^\tau\frac{Df}{D\tau} \\ &= \pm[1\pm 2f(p_\eta=0,\tau)]\frac{me\tau}{\pi}\mathrm{sign}[E(\tau)]\ln\left[1\pm\exp\left(-\frac{\pi m^2}{|eE(\tau)|}\right)\right]. \end{aligned} \tag{5.8}$$

Assuming that at $\tau=\tau_i$ there are no particles, the solution of (5.5) along the characteristic curves

$$\frac{dp_\eta}{d\tau} = eF_{\eta\tau}(\tau) \tag{5.9}$$

is

$$\begin{aligned} f(p_\eta,\tau) &= \pm\int_{\tau_i}^{\tau} d\tau'\,[1\pm 2f(p_\eta=0,\tau')]\,e\tau'|E(\tau')| \\ &\quad\times\ln\left[1\pm\exp\left(-\frac{\pi m^2}{e|E(\tau')|}\right)\right]\delta(p_\eta - eA_\eta(\tau') + eA_\eta(\tau)). \end{aligned} \tag{5.10}$$

Thus the system (5.5)–(5.6) reduces to [2]

$$-\tau\frac{dE}{d\tau} = \pm\frac{e^2}{\pi m^2}\int_{\tau_i}^{\tau} d\tau'[1\pm 2f(p_\eta=0,\tau')]\,\frac{A(\tau')-A(\tau)}{\sqrt{[A(\tau')-A(\tau)]^2+\tau^2}}\tau'$$

[2] A related derivation (but without the generality and fully covariance of the present one) for this model in terms of different variables can be found in [5, 6, 7].

$$\times |E(\tau')| \ln \left[1 \pm \exp\left(-\frac{\pi}{|E(\tau')|}\right)\right]$$
$$\pm[1 \pm 2f(p_\eta = 0, \tau)] \frac{\tau e^2}{\pi m^2} \,\mathrm{sign}\,(E(\tau)) \ln \left[1 + \exp\left(-\frac{\pi}{|E(\tau)|}\right)\right]. \tag{5.11}$$

In the above expression and in the following we introduce the dimensionless variables.

$$m\tau \to \tau\,, \qquad eA \to A\,, \qquad \frac{eE}{m^2} \to E\,, \qquad \frac{ej_\eta}{m^2} \to j_\eta\,, \tag{5.12}$$

For scalar particles we carry out the calculations in terms of the conformal proper time u (see Appendix C). This gives us more control over the physics at very early times, allowing us to choose a well-behaved initial adiabatic vacuum, corresponding to initial conditions

$$\begin{aligned} W_{\mathbf{k}}(u_0) &= w_{\mathbf{k}}(u_0), \\ \dot{W}_{\mathbf{k}}(u_0) &= \dot{w}_{\mathbf{k}}(u_0) \end{aligned} \tag{5.13}$$

It is worth mentioning that the adiabatic vacuum in terms of u is not identical to the adiabatic vacuum in terms of τ; they are related by a Bogolyubov transformation. The variable u is regular and improves numerical stability near the singular point $\tau = 0$.

For an initial adiabatic vacuum state the renormalized Maxwell equation is given by

$$-\frac{dE}{du} = \frac{e_R^2/m^2}{1 - e_R^2 \delta e^2} \int_{-\infty}^{\infty} \frac{dk_\eta}{2\pi} (k_\eta - A) \left[\frac{1}{W_{k_\eta}(u)} - \frac{1}{w_{k_\eta}(u)}\right], \tag{5.14}$$

where $\delta e^2 = (12\pi m^2)^{-1}$. Equations (5.14) and (C.7) define the numerical problem for the boson case. In solving (5.14) and (C.7) we discretize the momentum variable in a box with periodic boundary conditions, $k_\eta \to \pm 2\pi n/L$ where $L = 500$ and n ranges from 1 to 3×10^4. The time step in u was taken to be 5×10^{-4}.

To compare the Boltzmann-Vlasov phenomenological model to the above semiclassical system, Eq. (5.11) is written in terms of the conformal proper time variable u and becomes

$$-\frac{dE}{du} = \pm\frac{e^2}{\pi m^2}\int_{u_i}^{u} du'[1 \pm 2f(p_\eta = 0, u')]\frac{A(u') - A(u)}{\sqrt{[A(u') - A(u)]^2 + e^{2u}}}e^{2u'} \times |E(u')| \ln\left[1 \pm \exp\left(-\frac{\pi}{|E(u')|}\right)\right] \pm[1 \pm 2f(p_\eta = 0, u)]\frac{e^u e^2}{\pi m^2}\,\mathrm{sign}\,(E(u)) \ln\left[1 \pm \exp\left(-\frac{\pi}{|E(u)|}\right)\right], \tag{5.15}$$

where $dA/du = -e^{2u}E$.

In the fermion problem we perform the simulations in terms of τ, and in this case the semiclassical problem is defined by Equation (2.47) and by the Maxwell equation

$$\tau\frac{dE(\tau)}{d\tau} = -\frac{2(e_R^2/m^2)}{1 - e_R^2\delta e^2}\sum_{s=1}^{2}\int\frac{dk_\eta}{2\pi}\lambda_s|f_{\mathbf{k}s}^{+}|^2. \tag{5.16}$$

In this $1+1$ dimensions problem $\delta e^2 = (6\pi m^2)^{-1}$, $\lambda_1 = 1$, and $\lambda_2 = -1$. The time step in τ was taken to be 0.0005 with the momentum grid the same as for the scalar problem.

Figs. 1 and 2 summarize the results of the numerical simulations of charged scalar particles in $1+1$ dimensions. In Fig. 1 we show the time evolution of $A(u)$, $E(u)$ and $j_\eta(u)$ for the case of $E(u=-2) = 4$ and $e^2/m^2 = 1$. We see that in the first oscillation the electric field decays quite strongly, in contrast with the spatially homogeneous case [19]. In the latter the degradation of the electric field results from particle production only. In this inhomogeneous problem our system expands, and hence the initial electromagnetic energy density is reduced due to the particle production and this expansion. This degradation due to the expansion can be inferred by solving a classical system of particles and antiparticles that interact with a proper-time dependent electric field without a particle production source term. We also note that the larger the initial field, the smaller the period of oscillations.

In Fig. 2 we compare the time evolution of the phenomenological model (dashed curve) with the result of the semiclassical calculation (solid curve). Initial conditions were fixed at $u = -2$ [Figs. 2(a)–2(b)] and at $u = 0$

[Figs. 2(c)–2(d)]. We see that there is good agreement between the semiclassical solution and the Boltzmann-Vlasov model. This agreement also holds for different values of the initial electric field and for different coupling constants. The Bose enhancement increases the frequency, and hence a better agreement is achieved, as expected. We conclude that the WKB formula with Bose enhancement is a suitable source term for the boost-invariant problem. It is worth mentioning that at very early times of the evolution (before $\tau = 1$) the particle production is negligible and the electric field falls very slowly, as can bee seen in Figs. 2(c)–2(d).

In Figs. 3 through 9 we present the numerical results for fermions in $1+1$ dimensions. In Fig. 3 we compare the time evolution of the Boltzmann-Vlasov equation (dashed curve) with the results of the semiclassical calculation (solid curve), where the initial conditions were fixed at $\tau = 1$. All succeeding figures refer to these same initial conditions. In Fig. 4 we present the time evolution for $\tau\epsilon$ where $\epsilon = T_{\tau\tau}$. We see that at large τ this quantity oscillates around a fixed value. In Fig. 5 we show the time evolution of p/ϵ where $p\tau^2 = T_{\eta\eta}$. In this lowest order calculation there is no true dissipation and no particular equation of state emerges from the time evolution, although there is some indication that p approaches zero faster than ϵ. In Fig. 6 we present the evolution of the particle density $dN/d\eta$. After a short time (of order $\tau = 15$) the particle density reaches a plateau which doesn't change much in the subsequent evolution. This is consistent with the fact that the Schwinger particle creation mechanism turns off rapidly as the electric field decreases. In Fig. 7 we present the time evolution of $\tau\epsilon/(dN/d\eta)$. One can see that at large τ there is some indication that this ratio approaches the value of the mass (we choose $m = 1$), which agrees with the prediction of the hydrodynamic model discussed in Section 4. [see Eq. (4.29)]. This lends support to the idea that the pion spectrum can be calculated using (4.28). Defining the Boltzmann entropy by (A.10) of Appendix A, we plot τs as a function of τ in Fig. 8. Notice that this quantity is roughly constant after $\tau \approx 20$, by which time particle production has nearly ceased. [Compare Fig. 6.] This agrees with the result expected from the hydrodynamic point of view, eq. (4.23). Finally, in Fig. 9, we plot the effective "temperature," defined by the hydrodynamic relation (4.21), but using the Boltzmann entropy of Fig. 8.

In conclusion, the present results using boost-invariant coordinates fall into line with previous studies [19, 20] of boson and fermion pair production from an electric field with back-reaction. The renormalized field-theory calculation is tractable, yielding oscillatory behavior for a relativistic plasma which can also be well described by means of a classical transport equation with a source term derived from the Schwinger mechanism modified by Bose enhancement or Pauli blocking. For the boost-invariant case the electric field decays much more rapidly than for cartesian coordinates, where the sole decay mechanism is transfer of energy to the produced pairs. The ability to use the transport-equation approximation for boost-invariant pair production justifies in part the use that has been made of this method of description in past studies of the production of the quark-gluon plasma, and opens the way for further applications in the future.

A The particle spectrum of fermions

We present here the formulae for direct calculation of the fermion particle spectrum in the adiabatic method. For further discussion see [20, 21].

During particle production particle number is not conserved nor even uniquely defined. In terms of the adiabatic expansion of Sections 3 and 4, however, a natural definition of an interpolating particle-number operator suggests itself. One may simply expand the field in terms of the *time-dependent* creation and annihilation operators of the lowest order adiabatic vacuum,

$$\Psi(x) = \int [d\mathbf{k}] \sum_s [a_s(\mathbf{k};\tau) y^+_{\mathbf{k}s}(\tau) e^{i\mathbf{k}\cdot\mathbf{x}} + c^\dagger_s(-\mathbf{k};\tau) y^-_{-\mathbf{k}s}(\tau) e^{-i\mathbf{k}\cdot\mathbf{x}}]. \tag{A.1}$$

where

$$y^\pm_{\mathbf{k}s} = \left[-\gamma^0 \left(\frac{d}{d\tau} + \frac{1}{2\tau}\right) - i\gamma_\perp \cdot \mathbf{k}_\perp - i\gamma^3 \pi_\eta + m\right] \chi_s \frac{g^\pm_{\mathbf{k}s}}{\sqrt{\tau}}\,, \tag{A.2}$$

analogously to eq. (2.44) of the text, but in which the exact mode functions $f^\pm_{\mathbf{k}s}$ obeying (2.47) are replaced by lowest order adiabatic mode functions $g^\pm_{\mathbf{k}s}$. The positive frequency adiabatic mode function is given explicitly by substituting $\omega_\mathbf{k}(\tau)$ for $\Omega_\mathbf{k}(\tau)$ in the expressions (2.51) and (2.54) for $f^\pm_{\mathbf{k}s}$ in the text.

The adiabatic basis functions and the corresponding Fock space particle annihilation and creation operators, $a_s(\mathbf{k};\tau)$ and $c^\dagger_s(-\mathbf{k};\tau)$ defined in this way are related to those defined in (2.42) by a time-dependent Bogoliubov transformation. This transformation is easily found by using the Dirac inner product,

$$(u, v) \equiv \int d\Sigma^\mu \bar{u} \tilde{\gamma}_\mu v = \int d^d\mathbf{x} \sqrt{-g}\; u^\dagger v\,. \tag{A.3}$$

Indeed by substituting the two expansions of the field operator in terms of the two different bases (2.42) and (A.1) we find

$$a_s(\mathbf{k};\tau) = (y^+_{\mathbf{k}s}(\tau) e^{i\mathbf{k}\cdot\mathbf{x}}, \Psi) = \alpha(\mathbf{k}s;\tau) b_s(\mathbf{k}) + \beta^*(\mathbf{k}s;\tau) d^\dagger_s(-\mathbf{k})\,, \tag{A.4}$$

with

$$\alpha(\mathbf{k}s;\tau) = 2\Big\{\frac{dg^{+*}_{\mathbf{k}s}}{d\tau}\frac{df^+_{\mathbf{k}s}}{d\tau} + i\lambda_s \pi_\eta \Big(g^{+*}_{\mathbf{k}s} \frac{\overleftrightarrow{d}}{d\tau} f^+_{\mathbf{k}s}\Big) + \omega^2_\mathbf{k} g^{+*}_{\mathbf{k}s} f^+_{\mathbf{k}s}\Big\}\,. \tag{A.5}$$

Squaring this expression and using (2.51) and its analog for the adiabatic mode function $g^+_{\mathbf{k}s}$, we arrive at

$$
\begin{aligned}
|\beta(\mathbf{k}s;\tau)|^2 &= 1 - |\alpha(\mathbf{k}s;\tau)|^2 \\
&= 4|f^+_{\mathbf{k}s}|^2 |g^+_{\mathbf{k}s}|^2 (\omega^2_{\mathbf{k}} - \pi^2_\eta) \\
&\qquad \times \Big\{ (\Omega_{\mathbf{k}s} - \omega_{\mathbf{k}})^2 + \Big[\frac{(\dot{\Omega}_{\mathbf{k}s} + \lambda_s \dot{\pi}_\eta)}{2\Omega_{\mathbf{k}s}} - \frac{(\dot{\omega}_{\mathbf{k}} + \lambda_s \dot{\pi}_\eta)}{2\omega_{\mathbf{k}}} \Big]^2 \Big\} .
\end{aligned} \tag{A.6}
$$

The expectation value of the number operator with respect to the adiabatic Fock space operators in (A.1) is then simply the sum over $s = 1, 2$ or $s = 3, 4$ of

$$
\begin{aligned}
N(\mathbf{k}s, \tau) &= N_+(\mathbf{k}s)|\alpha(\mathbf{k}s;\tau)|^2 + (1 - N_-(\mathbf{k}s))|\beta(\mathbf{k}s;\tau)|^2 \\
&+ 2\mathrm{Re}\{\alpha(\mathbf{k}s;\tau)\beta(\mathbf{k}s;\tau)F(\mathbf{k}s)\}.
\end{aligned} \tag{A.7}
$$

For initial conditions which correspond to the adiabatic vacuum,

$$
\begin{aligned}
\Omega_{\mathbf{k}s}(\tau_0) &= \omega_{\mathbf{k}}(\tau_0), \\
\dot{\Omega}_{\mathbf{k}s}(\tau_0) &= \dot{\omega}_{\mathbf{k}}(\tau_0)
\end{aligned} \tag{A.8}
$$

one can choose $N_\pm(\mathbf{k}s) = F(\mathbf{k}s) = 0$, thus $N_\pm(\mathbf{k}s, \tau) = |\beta(\mathbf{k}s;\tau)|^2$. Hence, the time-dependent particle number defined by (A.7) with (A.6) has the property of starting at zero at $\tau = 0$ if the initial state is the adiabatic vacuum state. At late times, when electric fields go to zero it approaches the usual out-state number operator. Thus we can identify the phase-space density as

$$
\tilde{f}(k_\eta, \mathbf{k}_\perp, \tau) \equiv \frac{dN}{d\eta\, dk_\eta\, d\mathbf{k}_\perp d\mathbf{x}_\perp} = |\beta(k_\eta, \mathbf{k}_\perp, \tau)|^2. \tag{A.9}
$$

The quantity defined in this way from first principles of the microscopic quantum theory agrees quite well (after coarse graining) with the single particle distribution function $f(\mathbf{p}, \tau)$, obtained by solving the Boltzmann-Vlasov eq. (5.1). From the single particle distribution one can construct the Boltzmann entropy density in the comoving frame,

$$
s(\tau) = -\frac{1}{\tau} \int \frac{dk_\eta\, d\mathbf{k}_\perp}{(2\pi)^3} \{ \tilde{f} \ln \tilde{f} + (1 - \tilde{f}) \ln(1 - \tilde{f}) \} \ . \tag{A.10}
$$

If $\tilde{f}$ approaches a Fermi-Dirac equilibrium distribution at late τ, then this Boltzmann entropy will agree with the quantity (4.24) defined by the energy-momentum tensor in Section 4.

B Fluid rapidity distribution and particle rapidity distribution

In hydrodynamical models one has a purely phenomenological description in terms of the collective variables—energy density, pressure and hydrodynamic four-velocity. Using a criterion for hadronization such as those described by Landau [14] and by Cooper, Frye, and Schonberg [15] one determines the particle spectra by making a further assumption that at break-up the fluid velocity is equal to the particle velocity. In our field theory treatment no such further assumption is needed as long as boost invariance holds and we determine the particle spectrum along a surface of constant τ. In this appendix we will show the equivalence $dN/d\eta = dN/dy$.

One ingredient in the proof is the fact that the transformation of coordinates to the (η, τ) system is a transformation to a local frame that moves with constant velocity $\tanh \eta$ with respect to the Minkowski center-of-mass frame, i.e., the comoving frame is not accelerated with respect to the Minkowski frame. Because of this, the total number of particles counted in that frame is the same as the number of particles in the Minkowski frame of reference. The second ingredient is that in order for the phase-space volume to be preserved under our coordinate transformation, we need to ensure that the transformation in phase space is canonical in the classical sense of preserving Poisson brackets.

Consider then the coordinate transformation

$$\begin{aligned} \tau &= (t^2 - z^2)^{1/2} \qquad \eta = \frac{1}{2} \ln \left(\frac{t+z}{t-z} \right) \\ p_\tau &= Et/\tau - pz/\tau \quad p_\eta = -Ez + tp. \end{aligned} \tag{B.1}$$

The Poisson bracket is defined as

$$\{A, B\} = \frac{\partial A}{\partial p} \frac{\partial B}{\partial x} - \frac{\partial A}{\partial E} \frac{\partial B}{\partial t} - \frac{\partial B}{\partial p} \frac{\partial A}{\partial x} + \frac{\partial A}{\partial t} \frac{\partial B}{\partial E}\,. \tag{B.2}$$

The Poisson brackets of these quantities are

$$\begin{aligned} \{\tau, \eta\} = 0, &\quad \{p_\eta, p_\tau\} = 0, \\ \{p_\tau, \tau\} = -1, &\quad \{p_\eta, \eta\} = 1. \end{aligned} \tag{B.3}$$

We see that the above transformation is canonical.

The phase-space density of particles can be derived as shown in Appendix A, and it is found to be η-independent. In order to obtain the rapidity distribution, we change variables from (η, k_η) to (z, y), where y is the particle rapidity,

$$y = \frac{1}{2} \ln \left(\frac{E + k_z}{E - k_z} \right) . \tag{B.4}$$

Thus we have

$$\frac{dN}{d\eta\, dk_\eta\ d\mathbf{k}_\perp d\mathbf{x}_\perp} \quad = \quad J \, \frac{dN}{dy\, dz\, d\mathbf{k}_\perp d\mathbf{x}_\perp} \,, \tag{B.5}$$

where the Jacobian is

$$J^{-1} = \begin{vmatrix} \partial k_\eta / \partial y & \partial k_\eta / \partial z \\ \partial \eta / \partial y & \partial \eta / \partial z \end{vmatrix} = \frac{\partial k_\eta}{\partial y} \frac{\partial \eta}{\partial z}. \tag{B.6}$$

p_τ and p_η can be rewritten as

$$\begin{aligned} p_\tau &= m_\perp \cosh(\eta - y) \\ p_\eta &= -\tau m_\perp \sinh(\eta - y) \,. \end{aligned} \tag{B.7}$$

The particle spectrum is calculated at a fixed value of τ, so $\eta = \sinh^{-1}(z/\tau) = \eta(z)$. Thus functionally at fixed τ we have

$$p_\eta + eA_\eta \equiv k_\eta = k_\eta(\eta(z) - y) \,. \tag{B.8}$$

The chain rule then gives

$$\left. \frac{\partial k_\eta}{\partial z} \right|_\tau = \frac{\partial k_\eta}{\partial \eta} \frac{\partial \eta}{\partial z} = - \frac{\partial k_\eta}{\partial y} \frac{\partial \eta}{\partial z} \,. \tag{B.9}$$

At constant τ, then, $|J| = dz/dk_\eta$, which leads to the desired result

$$\frac{dN}{dy} = \frac{dN}{d\eta} \,. \tag{B.10}$$

Since the right-hand side of (A.9) is η independent, $dN/d\eta$ is flat in η. From (B.10) we conclude that the distribution dN/dy is flat, as expected.

C Scalar electrodynamics in conformal coordinates

Since $\tau = 0$ is a singular point of our equations, we find it convenient to introduce the conformal time coordinate u via[3]

$$m\tau = e^u \,. \tag{C.1}$$

In (1+1) dimensions the line element reads

$$ds^2 = -dt^2 + dz^2 = \frac{e^{2u}}{m^2}(-du^2 + d\eta^2)\,. \tag{C.2}$$

The transformation from the Minkowski t, z coordinates to the Kasner u, η coordinates is conformal. We shall refer to u as the conformal proper time.[4]

Instead of expanding the field χ as in (2.19) we expand the field ϕ with the mode functions $g_k = f_k/\sqrt{\tau}$, which satisfy

$$\frac{d^2 g_k}{d\tau^2} + \frac{1}{\tau}\frac{dg_k}{d\tau} + \left[m_\perp^2 + \frac{(k_\eta - eA(\tau))^2}{\tau^2}\right] g_k(\tau) = 0\,, \tag{C.3}$$

where $m_\perp^2 \equiv k_\perp^2 + m^2$. In terms of u the mode equation is

$$\frac{d^2 g_k}{du^2} + w_k^2(u) g_k(u) = 0\,, \tag{C.4}$$

where

$$w_k^2(u) \equiv \frac{m_\perp^2}{m^2} e^{2u} + (k_\eta - eA(u))^2\,. \tag{C.5}$$

We parametrize g_k in a WKB-like ansatz,

$$g_k(u) = \frac{1}{\sqrt{2W_k(u)}} \exp\left(-i \int^u W_k(u') du'\right)\,, \tag{C.6}$$

[3]In the radial Schrodinger equation a singularity at $r = 0$ prevents straightforward application of the WKB method. The transformation $r = r_0 \exp u$ maps the singularity from the origin to $-\infty$ and enables one to use the WKB approximation in terms of the new variable u [26]. Our situation is different, because in the vicinity of the singular point $\tau = 0$ we can still use the adiabatic expansion in terms of τ; the variable u is still helpful in avoiding numerical difficulties.

[4]For a general Kasner metric the conformal time is defined as $\eta_{conf} \equiv \int^\tau [(-g)^{1/2}]^{-1/3}$, where g is the determinant of (2.5).

and again the real mode functions W_k satisfy the differential equation

$$\frac{1}{2}\frac{\ddot{W}_k}{W_k} - \frac{3}{4}\frac{\dot{W}_k^2}{W_k^2} + W_k^2 = w_k^2(u)\,, \tag{C.7}$$

where the dot now denotes differentiation with respect to u. The Maxwell equation for an initial vacuum state is now

$$-\frac{dE}{du} = j_\eta \tag{C.8}$$

or

$$e^{-2u}\left(\frac{d^2A_\eta}{du^2} - 2\frac{dA_\eta}{du}\right) = e\int[d\mathbf{k}]\,\frac{k_\eta - eA_\eta(u)}{W_k(u)}\,. \tag{C.9}$$

Performing the adiabatic expansion of $W_k(u)$ for large k we find

$$e^{-2u}\left(\frac{d^2A_\eta}{du^2} - 2\frac{dA}{du}\right) = -e^{-2u}\left(\frac{d^2A_\eta}{du^2} - 2\frac{dA_\eta}{du}\right)e^2\delta e^2 + (\text{finite terms})\,, \tag{C.10}$$

where

$$\delta e^2 = \frac{1}{24\pi^2}\ln\Lambda/m \tag{C.11}$$

in (3+1) dimensions, and

$$\delta e^2 = \frac{1}{12\pi m^2} \tag{C.12}$$

in (1+1) dimensions, as expected. Note that the renormalization procedure introduces no additional interactions, in contrast to (3.11) in (η, τ) coordinates.

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References

[1] T. S. Biró, H. B. Nielsen, and J. Knoll, Nucl. Phys. **B245**, 449 (1984).

[2] F. Sauter, Z. Phys. **69**, 742 (1931).

[3] W. Heisenberg and H. Euler, Z. Phys. **98**, 714 (1936).

[4] J. Schwinger, Phys. Rev. **82**, 664 (1951).

[5] A. Białas and W. Czyż, Phys. Rev. D **30**, 2371 (1984); *ibid.* **31**, 198 (1985); Z. Phys. **C28**, 255 (1985); Nucl. Phys. **B267**, 242 (1985); Acta Phys. Pol. B **17**, 635 (1986).

[6] A. Białas, W. Czyż, A. Dyrek, and W. Florkowski, Nucl. Phys. **B296**, 611 (1988).

[7] K. Kajantie and T. Matsui, Phys. Lett. **164B**, 373 (1985).

[8] G. Gatoff, A. K. Kerman, and T. Matsui, Phys. Rev. D **36**, 114 (1987).

[9] R.P. Feynman, *Photon-Hadron Interactions* (W. A. Benjamin, Reading, Mass., 1972).

[10] J. Bjorken, in *Current Induced Reactions*, edited by J. G. Korner, G. Kramer, and D. Schildknecht (Springer-Verlag, Berlin, 1976), p. 93.

[11] A. Casher, J. Kogut, and L. Susskind, Phys. Rev. D **10**, 732 (1974).

[12] J. D. Bjorken, Phys. Rev. D**27**, 140 (1983).

[13] B. Andersson, G. Gustafson, G. Ingelman, and T. Sjostrand, Phys. Rep. **97**, 31 (1983).

[14] L. D. Landau, Izv. Akad. Nauk. SSSR (Ser. Fiz.) **17**, 51 (1953), translated in *Collected papers of L.D. Landau,* edited by D. ter Haar (Gordon and Breach, New York, 1965), p. 569.

[15] F. Cooper, G. Frye and E. Schonberg, Phys. Rev. **D11**, 192 (1975).

[16] F. E. Low, Phys. Rev. D **12**, 163 (1975).

[17] S. Nussinov, Phys. Rev. Lett. **34**, 1296 (1975).

[18] F. Cooper and E. Mottola, Phys. Rev. D **40**, 456 (1989).

[19] Y. Kluger, J. M. Eisenberg, B. Svetitsky, F. Cooper, and E. Mottola, Phys. Rev. Lett. **67**, 2427 (1991).

[20] Y. Kluger, J. M. Eisenberg, B. Svetitsky, F. Cooper, and E. Mottola, Phys. Rev. D **45**, 4659 (1992).

[21] F. Cooper, in *Particle Production in Highly-Excited Matter, Proc. NATO Advanced Study Institute, Il Ciocco, Italy, July, 1992*, edited by H. Gutbrod and J. Rafelski, to be published; Y. Kluger, J. M. Eisenberg, B. Svetitsky, Tel Aviv Preprint TAUP 2006-92, November, 1992, to be published in Int. J. Mod. Phys. E.

[22] J. Schwinger, J. Math. Phys. **2**, 407 (1961); L.V. Keldysh, Zh. Eksp. Teor. Fiz. (U.S.S.R.) **47**, 1515 (1964) [Sov. Phys. JETP **20**, 1018 (1965)]; K.-C. Chou, Z.-B. Su, B.-L. Hao, and L. Yu, Phys. Rep. **118**, 1 (1985).

[23] N. D. Birrell and P. C. W. Davies *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, 1982).

[24] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity* (Wiley, New York, 1972).

[25] L. Parker, Phys. Rev. D **3**, 346 (1971).

[26] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill, New York, 1953).

[27] M. S. Marinov and V. S. Popov, Fortsch. Phys. **25**, 373 (1977).

[28] B. S. DeWitt, Phys. Rep. **19C**, 295 (1975)

FIGURE CAPTIONS

FIGURE 1. Conformal proper time evolution of (a) the gauge field $A(u)$, (b) electric field $E(u)$, and (c) current $\jmath_\eta(u)$, for scalar particles in dimensionless units (5.12). The initial conditions are that of adiabatic vacuum with respect to conformal u time at $u = -2$ with initial electric field $E(u = -2) = 4.0$

FIGURE 2. Conformal proper time evolution of electric field (a) $E(u)$, and (b) scalar particle current $\jmath_\eta(u)$ with the same initial conditions as in Fig. 1 (solid lines) compared to solution of the Boltzmann-Vlasov equation (dashed line). (c) and (d) are the same as (a) and (b) but for initial adiabatic vacuum conditions at $u = 0$.

FIGURE 3. Proper time evolution of the system of (a) electric field $E(\tau)$, and (b) fermion current $\jmath_\eta(\tau)$, for initial conditions at $\tau = 1$ with initial electric field $E(\tau = 1) = 4.0$

FIGURE 4. Proper time evolution of $\tau\epsilon(\tau)$ for fermions.

FIGURE 5. Proper time evolution of p/ϵ for fermions.

FIGURE 6. Proper time evolution of $dN/d\eta$ for fermions.

FIGURE 7. Proper time evolution of $\tau\epsilon/(dN/d\eta)$ for fermions.

FIGURE 8. Proper time evolution of Boltzmann entropy density s, multiplied by τ for fermions.

FIGURE 9. Proper time evolution of the effective hydrodynamical “temperature” for fermions.