

# Lattice gauge theory with baryons at strong coupling

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## Abstract

We study the effective Hamiltonian for strong-coupling lattice QCD in the case of non-zero baryon density. In leading order the effective Hamiltonian is a generalized antiferromagnet. For naive fermions, the symmetry is  $U(4N_f)$  and the spins belong to a representation that depends on the local baryon number. Next-nearest neighbor ( $nnn$ ) terms in the Hamiltonian break the symmetry to  $U(N_f) \times U(N_f)$ . We transform the quantum problem to a Euclidean sigma model which we analyze in a  $1/N_c$  expansion. In the vacuum sector we recover spontaneous breaking of chiral symmetry for the nearest-neighbor and  $nnn$  theories. For non-zero baryon density we study the nearest-neighbor theory only, and show that the pattern of spontaneous symmetry breaking depends on the baryon density.

## I. INTRODUCTION

The study of quantum chromodynamics at high density is almost as old as the theory itself [1]. In recent years the field has attracted wide interest in the wake of a revival of the idea of color superconductivity (CSC) [2, 3]. The stimulus for this revival was the observation [4, 5] that the instanton-induced quark–quark interaction can be much stronger than that induced by simple one-gluon exchange, and can thus give a transition temperature on the order of 100 MeV. Subsequent work [6] showed that the perturbative color-magnetic interaction also gives rise to a strong pairing interaction.

These and other dynamical considerations [7] underlie a picture of the ground state of high-density QCD in which the  $SU(3)$  gauge symmetry is spontaneously broken by a BCS-like condensate. The details of the breaking, which include both the Higgs (or Meissner) effect and the rearrangement of global symmetries and Goldstone bosons, depend on quark masses, chemical potentials, and temperature. Prominent in the list of possibilities are those of color-flavor locking in three-flavor QCD [8] and crystalline superconductivity—with broken translation invariance—when there are two flavors with different densities [9]. For a review see [10].

As noted, CSC at high density is so far a prediction of weak-coupling analysis. One expects the coupling to become weak only at high densities, and in fact it turns out that reliable calculations demand extremely high densities [11]. The use of weak-coupling methods to make predictions for moderate densities is thus not an application of QCD, but of a model based on it. It is imperative to confirm these predictions by non-perturbative methods. Standard lattice Monte Carlo methods, unfortunately, fall afoul of well-known technical problems when the chemical potential is made nonzero, although we do note remarkable progress made recently in the small- $\mu$  regime [12, 13].

In this paper we initiate a study of high-density quark matter based on lattice QCD in the strong-coupling limit.<sup>1</sup> We work in the Hamiltonian formalism, which is more amenable than the Euclidean formalism to strong-coupling perturbation theory and to qualitative study of the ensuing effective theory [15–18]. The fermion kinetic Hamiltonian is a perturbation that mixes the zero-flux states that are the ground-state sector of the electric term in the gauge Hamiltonian. In second order, it moves color-singlet fermion pairs around the lattice; the effective Hamiltonian for these pairs is a generalized antiferromagnet, with spin operators constructed of fermion bilinears.

We depart from studies of the vacuum by allowing a background baryon density, which is perforce static in second order in perturbation theory. Our aim at this stage is to discover the ground state of the theory with this background. In third order (when  $N_c = 3$ ) the baryons become dynamical; we display the effective Hamiltonian but make no attempt to treat it.

The symmetry group of the effective antiferromagnet is the same as the global symmetry group of the original gauge theory. This depends on the formulation chosen for the lattice fermions. Following [16], we begin with naive, nearest-neighbor fermions, which suffer from species doubling [19] and possess a global  $U(4N_f)$  symmetry group that contains the ordinary chiral symmetries [as well as the axial  $U(1)$ ] as subgroups. We subsequently break the too-large symmetry group with next-nearest-neighbor ( $nnn$ ) couplings along the axes in the fermion hopping Hamiltonian. A glance at the menu of fermion formulations reveals the

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<sup>1</sup> An early discussion of our program, with early results, was given in [14].

reasons for our choice. Wilson fermions [20] have no chiral symmetry and make comparison of results to continuum CSC difficult if not impossible. Staggered fermions [21] likewise possess only a reduced axial symmetry while suffering a reduced doubling problem. The overlap action [22] is non-local in time and hence possesses no Hamiltonian; attempts [23] to construct an overlap Hamiltonian directly have not borne fruit. Finally, domain-wall fermions [24, 25] have been shown [18] to lose chiral symmetry and regain doubling when the coupling is strong.

As we discuss below, while the  $nnn$  theory still exhibits doubling in the free fermion spectrum, we are not interested in the perturbative fermion propagator but in the spectrum of the confining theory. We take it as a positive sign that the unbroken symmetry is now  $U(N_f) \times U(N_f)$ . This symmetry is what we want for the continuum theory, except for the axial  $U(1)$ . The latter can still be broken by hand.<sup>2</sup>

Our emphasis on the global symmetries is a consequence of the fact that the gauge field is not present in the ground-state sector and does not reappear in strong-coupling perturbation theory. In other words, confinement is a *kinematic* feature of the theory, leaving no possibility of seeing the Higgs-Meissner effect directly. This is but an instance of confinement-Higgs duality, typical of gauge theories with matter fields in the fundamental representation [28]. Our aim is thus to identify the pattern of spontaneous breaking of global symmetries. For various values of  $N_c$  and  $N_f$ , this can be compared to weak-coupling results [29].

This paper is largely an exposition of formalism, along with partial results. We study the nearest-neighbor antiferromagnetic Hamiltonian, both with and without a uniform baryonic background density. We transform the quantum Hamiltonian into a path integral for a non-linear  $\sigma$  model, where the manifold of the  $\sigma$  field depends on the baryon background. We then investigate the limit of large  $N_c$  and show that the global  $U(4N_f)$  symmetry is indeed spontaneously broken.

Adding in the  $nnn$  couplings is a problem of vacuum alignment [33]. We do this in the vacuum sector and recover the result [16] that the  $U(N_f) \times U(N_f)$  chiral symmetry is broken to the vector  $U(N_f)$ . The analysis for the finite-density theory is more involved and we defer it to a future publication.

Other groups have recently studied the strong-coupling effective Hamiltonian for naive and Wilson fermions at non-zero chemical potential [30–32]. We differ from their approaches in eschewing mean field theory in favor of the exact transformation to the  $\sigma$  model, which is amenable to semiclassical treatment. As noted above, we base our program on  $nnn$  fermions; we also work at fixed baryon density.

Let us walk through the paper. We review in Sec. II the derivation of the effective Hamiltonian of lattice gauge theory in strong-coupling perturbation theory [16, 17]. The second-order Hamiltonian [ $O(1/g^2)$ ] is an antiferromagnet with  $U(4N_f)$  spins; the global symmetry group is  $U(4N_f)$  for the nearest-neighbor theory, broken to  $U(N_f) \times U(N_f)$  by  $nnn$  terms. The baryon number at each site determines the representation of  $U(4N_f)$  carried by the spin at that site. In second order, baryon number is static; it becomes mobile in the next order, where (for  $N_c = 3$ ) the new term in the effective Hamiltonian is a baryon hopping term.

The baryon operators responsible for the hopping are composite operators that do not obey canonical anti-commutation relations. If this were not the case, then the effective

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<sup>2</sup> The breaking of the naive fermions' symmetry by longer-range terms is a feature [16] of SLAC fermions [26] and also occurs if naive fermions are placed on a *bcc* lattice [27].

Hamiltonian in third order would strongly resemble that of the  $t$ - $J$  model [34],

$$H_{t-J} = -t \sum_{\langle ij \rangle_s} c_{is}^\dagger c_{js} + J \sum_{\langle ij \rangle} \left( \mathbf{S}_i \cdot \mathbf{S}_j - \frac{n_i n_j}{4} \right) + \mathcal{J}'. \quad (1.1)$$

Here  $c_{js}$  is an annihilation operator for an electron at site  $j$  with spin  $s$ , and the number operators  $n_i = c_i^\dagger c_i$  and spin operators  $\mathbf{S}_i = \frac{1}{2} c_i^\dagger \boldsymbol{\sigma} c_i$  are constructed from it. The added term  $\mathcal{J}'$  is a more complicated hopping and interaction term. The  $t$ - $J$  model describes a doped antiferromagnet; it arises as the strong-binding limit of Hubbard model, a popular model for itinerant magnetism and possibly for high- $T_c$  superconductivity. The model is not particularly tractable and, absent new theoretical developments, does not offer much hope for progress in our finite-density problem. It is nonetheless worth pondering the fact that a model connected, however tentatively, with superconductivity appears in a study of high-density nuclear matter.

In the remainder of this paper, we work only to  $O(1/g^2)$ , where the baryons are fixed in position. Motivated by the similarity of our Hamiltonian to the Heisenberg antiferromagnet, we apply condensed matter methods developed for that problem. Indeed, condensed matter physicists have generalized the  $SU(2)$ , spin-1/2 Heisenberg model to  $SU(N)$  in many representations [34–41], which corresponds to adding flavor and color degrees of freedom to the electrons.<sup>3</sup> These are exactly the generalizations needed for our effective Hamiltonian. With  $N_c$  colors and  $N$  (single-component) flavors, a site of the lattice can be constrained to contain a color-singlet combination of  $mN_c$  particles. The flavor indices of the spin then make up a representation of  $SU(N)$  whose Young diagram has  $N_c$  columns and  $m$  rows (see Fig. 1). We set

$$N = 4N_f \quad (1.2)$$

and the correspondence is complete (until we include  $nnn$  terms in the Hamiltonian).

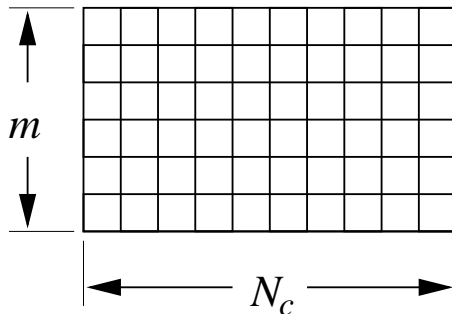


FIG. 1: The representation of  $U(4N_f)$  carried by the spin in the effective antiferromagnet.  $m$  is related to the baryon number at the site according to  $m = B + 2N_f$ , with  $|B| \leq 2N_f$ .

In Sec. III we derive a  $\sigma$  model representation for the partition function of the antiferromagnet. Following Read and Sachdev [39], we employ spin coherent states [43] to define the  $\sigma$  field.  $N$  and  $m$  determine the target space of the  $\sigma$  model to be the symmetric space  $U(N)/[U(m) \times U(N - m)]$ ; the number of colors  $N_c$  becomes an overall coefficient of the

<sup>3</sup> We refer the reader to the paper by Read and Sachdev [39] for a survey, including a phase diagram in the  $(N, N_c)$  plane.

action.<sup>4</sup> As for the quantum Hamiltonian, the nearest-neighbor theory is symmetric under  $U(N)$  while the  $nnn$  terms break the symmetry to  $U(N_f) \times U(N_f)$  (while leaving the manifold unchanged).

The  $N_c$  multiplying the action invites a large- $N_c$  analysis, and in Sec. IV we study the vacuum sector, meaning zero baryon number, thereby. We return to an exercise proposed and solved by Smit [17], in generalizing the vacuum sector to allow baryon number  $\pm B$  on alternating sites; this means specifying conjugate representations of  $U(N)$  on alternating sites, with respectively  $m$  and  $N - m$  rows. As shown by Read and Sachdev [39], in this situation one can carry out an alternating  $U(N)$  rotation to convert the antiferromagnet into a ferromagnet with *identical* spins on alternating sites, and the classical ( $N_c = \infty$ ) analysis gives a homogeneous ground state. The result is, as one might expect, that  $U(N)$  is broken to  $U(m) \times U(N - m)$  in the classical vacuum; the ground state energy is independent of  $m$ . The  $1/N_c$  corrections to the energy do depend on  $m$ , however, and they select the self-conjugate  $m = N/2$  configuration (*i.e.*,  $B = 0$  everywhere) as the lowest-energy vacuum. Thus the true ground state breaks  $U(N) \rightarrow U(N/2) \times U(N/2)$ .<sup>5</sup> When we add  $nnn$  terms to the action as a perturbation, we find that the ground state breaks  $U(N_f) \times U(N_f)$  to the vector  $U(N_f)$ , as expected.

We turn to non-zero baryon density in Sec. V. We study homogeneous states, in which all sites carry the same representation of  $U(N)$ , with  $m > N/2$ . The classical vacuum of the  $\sigma$  model is more elusive than for the vacuum sector, since now there are identical manifolds on adjacent sites but the coupling is antiferromagnetic. We begin by studying the two-site problem, and we learn that when one of the classical spins is fixed then when the energy is minimized the other spin is still free to wander a submanifold of the original symmetric space. If we replicate this to the infinite lattice then we have a situation where the even spins, say, are fixed in direction while each odd spin wanders the submanifold, independent of the other odd spins. This means a ground state whose degeneracy is exponential in the volume, similar to some frustrated models or the antiferromagnetic Potts model [45]. The cure to this disease comes from the  $O(1/N_c)$  fluctuations, which couple the odd spins to each other and make them align. In the end we find that the  $U(N)$  symmetry is broken by the vacuum to  $U(2m - N) \times U(N - m) \times U(N - m)$ . Perturbing this ground state with the  $nnn$  terms is technically difficult, and we do not attempt it here despite its obvious physical interest.

We close with a brief summary and discussion. The  $O(1/N_c)$  calculation in the  $B \neq 0$  case is relegated to an appendix, as are other (but not all) technical details.

## II. THE EFFECTIVE HAMILTONIAN

For an  $SU(N_c)$  gauge theory with  $N_f$  flavors of fermions, we write the lattice Hamiltonian

$$H = H_E + H_U + H_F. \tag{2.1}$$

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<sup>4</sup> The inverse gauge coupling  $1/g^2$  multiplies the quantum Hamiltonian, and hence serves only to set the energy scale.

<sup>5</sup> This result was obtained by Smit using a Holstein-Primakoff transformation on the quantum Hamiltonian. We note in passing that the  $1/N_c$  calculation includes the effect of the time-derivative terms in the action that were dropped in the leading order, and these terms *do* remember the difference between the ferromagnet and the antiferromagnet.

Here  $H_E$  is the electric term, a sum over links ( $\mathbf{n}\mu$ ) of the  $SU(N_c)$  Casimir operator on each link,

$$H_E = \frac{1}{2}g^2 \sum_{\mathbf{n}\mu} E_{\mathbf{n}\mu}^2. \quad (2.2)$$

Next is the magnetic term, a sum over plaquettes,

$$H_U = \frac{1}{2g^2} \sum_p (N_c - \text{Tr } U_p). \quad (2.3)$$

Finally we have the fermion Hamiltonian,

$$H_F = -i \sum_{\mathbf{n}\mu} \psi_{\mathbf{n}}^{\dagger\alpha f} \alpha_\mu \sum_{j>0} D(j) \left( \prod_{k=0}^{j-1} U_{\mathbf{n}+k\hat{\mu},\mu} \right)_{\alpha\beta} \psi_{\mathbf{n}+j\hat{\mu}}^{\beta f} + h.c. \quad (2.4)$$

The fermion field  $\psi_{\mathbf{n}}^{\alpha f}$  carries color  $\alpha$  and flavor  $f$  at site  $\mathbf{n}$ . The function  $D(j)$  is a kernel that defines the lattice fermion derivative. It can yield a naive, nearest-neighbor action if  $D(j) = \frac{1}{2}\delta_{j,1}$ ; a long-range SLAC derivative [26] if  $D(j) = -(-1)^j/j$ ; or anything in between, such as a  $nnn$  action obtained by truncating the SLAC kernel to its  $j = 1, 2$  terms.

For  $g \gg 1$  the ground state of  $H$  is determined by  $H_E$  alone to be any state with zero electric field, whatever its fermion content,

$$|0\rangle|\chi\rangle_F = \left[ \prod_{\mathbf{n}\mu} |E_{\mathbf{n}\mu}^2 = 0\rangle \right] |\chi\rangle_F. \quad (2.5)$$

These states have energy  $\epsilon_0 = 0$  and are degenerate with respect to all the fermionic degrees of freedom. We consider perturbation theory in  $V = H_U + H_F$ . Both  $H_U$  and  $H_F$  are sums of operators that are strictly bounded, independent of  $g$  except for the explicit coefficient in  $H_U$ . We can dismiss first-order perturbations by noting that  $H_U$  and  $H_F$  are multilinear in link operators  $U$  and  $U^\dagger$ , which are raising/lowering operators for the electric field; thus there are no non-zero matrix elements within the zero-field sector.

We proceed to higher orders, and seek an effective Hamiltonian that acts in the zero-field sector [42]. Define  $P_0$  to be the projector onto the subspace of all the  $E = 0$  states. Then perturbation theory in  $V$  gives an effective Hamiltonian,

$$H_{\text{eff}} = P_0 V Q D V P_0 + P_0 V Q D V Q D V P_0 + \dots \quad (2.6)$$

Here  $Q = 1 - P_0$  projects onto the subspace orthogonal to the  $E = 0$  states; the operator  $D \equiv (\epsilon_0 - H_E)^{-1}$  supplies energy denominators, so that

$$QD = \sum_{E \neq 0 \text{ states}} |\lambda\rangle \frac{1}{\epsilon_0 - \epsilon_\lambda} \langle \lambda|. \quad (2.7)$$

The intermediate states  $|\lambda\rangle$  contain flux excitations. In second and third order the patterns of flux can only be strings of length  $j$  in the fundamental representation of the color group. Thus the energy denominators are

$$\epsilon_0 - \epsilon_\lambda = -\frac{1}{2}g^2 C_F |j|, \quad (2.8)$$

where  $C_F = (N_c^2 - 1)/(2N_c)$  is the quadratic Casimir of the fundamental representation of  $SU(N_c)$ .

The perturbations  $H_U$  and  $H_F$  are explicitly of  $O(1/g^2)$  and  $O(1)$ , respectively; each energy denominator gives a factor of  $1/g^2$ . Thus to  $O(1/g^4)$  we can forget about  $H_U$ . Our result to this order is

$$H_{\text{eff}} = P_0 H_F D H_F P_0 + P_0 H_F D H_F D H_F P_0. \quad (2.9)$$

Since  $H_F$  has no non-zero matrix elements within the  $E = 0$  sector, we have dispensed with  $Q$  in Eq. (2.9). The first term in Eq. (2.9) arises for any value of  $N_c$  and is  $O(1/g^2)$ ; the case  $N_c = 2$  must be treated carefully, but all cases  $N_c > 2$  are generic. The second term is special to  $N_c = 3$  and is  $O(1/g^4)$ .

### A. Second order: the antiferromagnet

We calculate explicitly the first term in  $H_{\text{eff}}$ . Each term in  $H_F$  creates a string of flux of length  $j$ , which must be destroyed by the conjugate term. Thus

$$H_{\text{eff}}^{(2)} = 2 \sum_{j>0} [-K(j)] \sum_{\mathbf{n}\mu} \left( \psi_{\mathbf{n}}^{\dagger\alpha f} \alpha_{\mu} \psi_{\mathbf{n}+j\hat{\mu}}^{\beta f} \right) \langle 0 | \left( \prod U \right)_{\alpha\beta} \left( \prod U^{\dagger} \right)_{\gamma\delta} | 0 \rangle \left( \psi_{\mathbf{n}+j\hat{\mu}}^{\dagger\gamma g} \alpha_{\mu} \psi_{\mathbf{n}}^{\delta g} \right), \quad (2.10)$$

where we define

$$K(j) = \frac{[D(j)]^2}{\frac{1}{2}g^2 C_F |j|} > 0. \quad (2.11)$$

The matrix element of the gauge fields yields  $\frac{1}{N_c} \delta_{\alpha\delta} \delta_{\beta\gamma}$ , independent of  $j$ .

As they appear in Eq. (2.10), each  $\psi^{\dagger}$  is next to a  $\psi$  on a different site. This invites a Fierz transformation on the product of fermion fields, which we write generally as

$$\left( \psi_i^{\dagger} \alpha_{\mu} \psi_j \right) \left( \psi_k^{\dagger} \alpha_{\mu} \psi_l \right) = \delta_{jk} \psi_i^{\dagger} \psi_l - \frac{1}{4} \sum_A s_A^{\mu} \left( \psi_i^{\dagger} \Gamma^A \psi_l \right) \left( \psi_k^{\dagger} \Gamma^A \psi_j \right). \quad (2.12)$$

Here  $i, j, k, l$  are combined site, flavor, and color indices, and we have assumed that  $k$  and  $l$  are always different while  $j$  and  $k$  might be equal [as in Eq. (2.10)]. The matrices  $\Gamma^A$  are the 16 Dirac matrices, normalized to  $(\Gamma^A)^2 = \mathbf{1}$ , and we have defined

$$s_A^{\mu} = \frac{1}{4} \text{Tr} \Gamma^A \alpha_{\mu} \Gamma^A \alpha_{\mu} = \pm 1. \quad (2.13)$$

This sign factor is  $\pm 1$  according to whether  $\Gamma^A$  commutes or anticommutes with  $\alpha_{\mu}$ ; it will be a constant companion in our calculations. As they appear in  $H_{\text{eff}}^{(2)}$ , the indices  $i, l$  are the same site and color but different flavors, and likewise  $j, k$ . Leaving the flavor indices explicit, we obtain

$$H_{\text{eff}}^{(2)} = \frac{1}{4N_c} \sum_{\substack{\mathbf{n}\mu \\ j \neq 0}} K(j) s_A^{\mu} \left( \psi^{\dagger f} \Gamma^A \psi^g \right)_{\mathbf{n}} \left( \psi^{\dagger g} \Gamma^A \psi^f \right)_{\mathbf{n}+j\hat{\mu}} - d \sum_{j \neq 0} K(j) \sum_{\mathbf{n}} \left( \psi^{\dagger f} \psi^f \right)_{\mathbf{n}}. \quad (2.14)$$

Each fermion bilinear in parentheses is a color singlet located at a given site. The second term contains the baryon density<sup>6</sup>  $B'_n = N_c^{-1} \psi_n^\dagger \psi_n$ , and the sum  $\sum_n B'_n$  is the total baryon number  $B'$ .

We now combine the Dirac indices with the flavor indices and write

$$\left(\psi_n^\dagger f \Gamma^A \psi_n^g\right) \left(\psi_n^\dagger g \Gamma^A \psi_n^f\right) = 8 \left(\psi_n^\dagger M^n \psi_n\right) \left(\psi_n^\dagger M^n \psi_n\right). \quad (2.15)$$

We have defined new matrices  $M^n$  as direct products of the  $4 \times 4$  Dirac matrices and the  $U(N_f)$  flavor generators,

$$M^n = \Gamma^A \otimes \lambda^a, \quad (2.16)$$

and we have normalized them conventionally according to

$$\text{Tr } M^n M^{n'} = \frac{1}{2} \delta^{nn'}. \quad (2.17)$$

The  $M^n$  generate a  $U(N)$  algebra, with  $N \equiv 4N_f$ .

An alternating flip

$$\psi_n \rightarrow \left[ \prod_\mu (\alpha_\mu)^{n_\mu} \right] \psi_n \quad (2.18)$$

(spin diagonalization [16]) removes the  $\alpha_\mu$  matrices from the odd- $j$  terms in  $H_F$ , and hence removes the sign factors  $s_A^\mu$  from the odd- $j$  terms in  $H_{\text{eff}}^{(2)}$ . We have finally

$$H_{\text{eff}}^{(2)} = \frac{2}{N_c} \sum_{n\mu j} K(j) \left(s_\eta^\mu\right)_{\text{even } j} \left(\psi_n^\dagger M^n \psi_n\right) \left(\psi_{n+j\hat{\mu}}^\dagger M^n \psi_{n+j\hat{\mu}}\right) - \left(dN_c \sum_j K(j)\right) B'. \quad (2.19)$$

The odd- $j$  terms are of the form  $\mathbf{M} \cdot \mathbf{M}$  which can be written in any basis for the  $U(4N_f)$  algebra. The even- $j$  terms, however, contain  $s_\eta^\mu$  which is defined only in the original basis (2.16).

## B. Single-site states

In the zero-field sector in which we work, Gauss' Law constrains the fermion state at each site to be a color singlet. The drained state  $|\text{dr}\rangle$ , with  $\psi_n^{\alpha f} |\text{dr}\rangle = 0$  for all  $(\alpha, f)$ , is the unique state with  $B' = 0$ . The other color singlet states may be generated by repeated application of the baryon creation operator,

$$b_{fg\dots}^\dagger = \epsilon_{\alpha\beta\dots} \psi_n^{\dagger\alpha f} \psi_n^{\dagger\beta g} \dots, \quad (2.20)$$

with  $N_c$  operators  $\psi$ . (Here and henceforth, the indices  $f, g, \dots$  combine the flavor and Dirac indices.)

As noted above, at each site  $\mathbf{n}$  the operators

$$Q_n^\eta = \psi_n^\dagger M^n \psi_n = \psi_n^{\dagger\alpha f} M_{fg}^\eta \psi_n^{\alpha g} \quad (2.21)$$

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<sup>6</sup> This baryon number is positive semidefinite, and is zero for the drained state (see below). The conventional baryon number  $B_n$  is zero in the half-filled state, and thus  $B'_n = B_n + 2N_f$ .



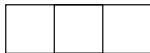


FIG. 2: Young diagram of the representation of  $U(4N_f)$  carried by the baryon operator

generate a  $U(N)$  algebra, with  $N = 4N_f$ . The drained state is obviously a singlet under this algebra. The creation operator  $b_{fg\dots}^\dagger$  is in the symmetric representation of  $U(N)$  with one row and  $N_c$  columns (see Fig. 2). Repeated application of  $b_{fg\dots}^\dagger$  to the drained state gives the state

$$|\chi\rangle = b^\dagger b^\dagger \dots |\text{dr}\rangle. \quad (2.22)$$

If there are  $m$  operators  $b^\dagger$ , then the state  $|\chi\rangle$  lies in the representation with  $N_c$  columns and  $m$  rows (see Fig. 1). Its baryon number is  $B'_n = m$ .

The second-order effective Hamiltonian  $H_{\text{eff}}^{(2)}$  preserves  $B'_n$ , the baryon number on each site. Thus any distribution of  $B'_n$  defines a sector within which  $H_{\text{eff}}^{(2)}$  is to be diagonalized. In other words, baryons constitute a fixed background in which to study “mesonic” dynamics. The baryon number at each site fixes the representation of  $U(N)$  at that site, which is the space of states in which the charges  $Q_n^\eta$  act.

### C. Global symmetries and doubling

The  $j = 1$  terms in Eq. (2.19) are of the form  $Q_n^\eta Q_{n+\hat{\mu}}^\eta$ , and they commute with the generators

$$Q^\eta = \sum_n Q_n^\eta \quad (2.23)$$

of a global  $U(N)$  symmetry group. This symmetry is in fact familiar from the lattice Hamiltonian of naive, nearest-neighbor fermions: Spin diagonalization of  $N_f$  naive Dirac fermions transforms the Hamiltonian into that of  $4N_f$  staggered fermions. In the weak coupling limit, there are in fact  $8N_f$  fermion flavors—the doubling problem. This doubling is partially reflected in the accidental  $U(4N_f)$  symmetry, which is intact in the  $g \rightarrow \infty$  limit and is respected by the effective Hamiltonian. Retaining terms in the fermion Hamiltonian (and thus in  $H_{\text{eff}}^{(2)}$ ) that involve odd separations  $j$  does not break this symmetry.

The Nielsen-Ninomiya theorem [19] guarantees that any fermion Hamiltonian of finite range will possess the full doubling problem. This is a statement, however, about weak coupling only, since the dispersion relation of free fermions is irrelevant if the coupling is strong and the fermions are confined. It is interesting that the accidental  $U(4N_f)$  symmetry nonetheless survives into strong coupling as a vestige of doubling.

The terms in Eq. (2.19) with *even*  $j$ , on the other hand, break the  $U(N)$  symmetry, as do even- $j$  terms in the original fermion Hamiltonian. It is easy to see via spin diagonalization, which leaves the even- $j$  terms unchanged, that the only generators left unbroken are the  $Q^\eta$  corresponding to

$$M^\eta = \mathbf{1} \otimes \lambda^a \quad \text{and} \quad \gamma_5 \otimes \lambda^a, \quad (2.24)$$

which form the  $U(N_f)_L \times U(N_f)_R$  chiral algebra. This of course makes no difference to the Nielsen-Ninomiya theorem, which will enforce 8-fold doubling in the perturbative propagator even without the  $U(4N_f)$  symmetry. If we are interested in the realization of the global symmetries of the continuum theory, though, we can study this lattice theory which has the

same symmetry. The simplest theory one may study is thus one containing nearest-neighbor and  $nnn$  terms. We shall proceed to discard terms with longer range; we shall begin with the nearest-neighbor theory, with its accidental doubling symmetry, and later break this symmetry to  $U(N_f)_L \times U(N_f)_R$  with the  $nnn$  terms.

Two essential differences will always remain between this lattice theory and the continuum theory. One is the presence of the axial  $U(1)$  symmetry on the lattice. This symmetry is exact, broken by no anomaly, and may make the drawing of conclusions for the continuum theory less than straightforward unless it is broken by hand. The other difference is the fact that the effective Hamiltonian for baryons (see below) is also a short-ranged hopping Hamiltonian. If the baryons were almost free, we would say that they are surely doubled like the original quarks. The fact that the simplicity of the hopping terms is only apparent, and that the baryons are still coupled strongly to mesonic excitations, offers the possibility that doubling may not return.

#### D. Third order: the baryon kinetic term

The third-order term in  $H_{\text{eff}}$ , which only exists in the case of  $N_c = 3$ , is calculated via

$$H_{\text{eff}}^{(3)} = P_0 V_F D V_F D V_F P_0. \quad (2.25)$$

For a single link, we have

$$\langle 0 | U_{\alpha\beta} U_{\gamma\delta} U_{\epsilon\zeta} | 0 \rangle = \frac{1}{6} \epsilon_{\alpha\gamma\epsilon} \epsilon_{\beta\delta\zeta}, \quad (2.26)$$

and the same can be proven for a chain of links,

$$\langle 0 | (\prod U)_{\alpha\beta} (\prod U)_{\gamma\delta} (\prod U)_{\epsilon\zeta} | 0 \rangle = \frac{1}{6} \epsilon_{\alpha\gamma\epsilon} \epsilon_{\beta\delta\zeta}. \quad (2.27)$$

Thus [ $f, g, \dots$  are here (temporarily) flavor indices],

$$H_{\text{eff}}^{(3)} = -i \sum_{j>0} \tilde{K}(j) \sum_{\mathbf{n}\mu} \left( \psi_{\mathbf{n}}^{\dagger f\alpha} \alpha_{\mu} \psi_{\mathbf{n}+j\hat{\mu}}^{f\beta} \right) \left( \psi_{\mathbf{n}}^{\dagger g\gamma} \alpha_{\mu} \psi_{\mathbf{n}+j\hat{\mu}}^{g\delta} \right) \left( \psi_{\mathbf{n}}^{\dagger h\epsilon} \alpha_{\mu} \psi_{\mathbf{n}+j\hat{\mu}}^{h\zeta} \right) \epsilon_{\alpha\gamma\epsilon} \epsilon_{\beta\delta\zeta} + h.c. \quad (2.28)$$

The kernel is

$$\tilde{K}(j) = \frac{(D(j))^3}{6 \left( \frac{1}{2} g^2 C_F |j| \right)^2}. \quad (2.29)$$

Again, spin diagonalization simplifies the odd- $j$  terms, but not the even- $j$  terms. The result is

$$H_{\text{eff}}^{(3)} = H_{\text{odd}}^{(3)} + H_{\text{even}}^{(3)}, \quad (2.30)$$

with

$$H_{\text{odd}}^{(3)} = -i \sum_{\substack{j>0 \\ j \text{ odd}}} \tilde{K}(j) \sum_{\mathbf{n}\mu} b_{\mathbf{n}}^{\dagger I} b_{\mathbf{n}+j\hat{\mu}}^I \eta_{\mu}(\mathbf{n}) + h.c., \quad (2.31)$$

where  $\eta_{\mu}(\mathbf{n})$  is the usual staggered-fermion sign factor, and

$$H_{\text{even}}^{(3)} = -i \sum_{\substack{j>0 \\ j \text{ even}}} \tilde{K}(j) \sum_{\mathbf{n}\mu} b_{\mathbf{n}}^{\dagger I} [\alpha_{\mu} \otimes \alpha_{\mu} \otimes \alpha_{\mu}]_{II'} b_{\mathbf{n}+j\hat{\mu}}^{I'} \zeta_{\mu}(\mathbf{n}) + h.c., \quad (2.32)$$

where  $\zeta_\mu(\mathbf{n}) = (-1)^{\sum_{\nu \neq \mu} n_\nu}$ . The baryon operators are

$$b^I = \epsilon_{\alpha\beta\gamma} \psi^{\alpha f} \psi^{\beta g} \psi^{\gamma h}, \quad (2.33)$$

where we have written  $I$  to represent the compound index  $\{fgh\}$ , taking values in the symmetric three-index representation of  $U(N)$  ( $f, g, \dots$  once more combine flavor and Dirac indices). The odd- $j$  part of  $H_{\text{eff}}^{(3)}$ , like that of  $H_{\text{eff}}^{(2)}$ , is symmetric under the  $U(N)$  doubling symmetry. The even- $j$  part breaks  $U(N)$  to  $U(N_f)_L \times U(N_f)_R$ .

$H_{\text{eff}}^{(3)}$  is a baryon hopping term. As mentioned in the Introduction, however, its simplicity is deceptive. The baryon operators  $b_{\mathbf{n}}^I$  are composite and hence do not obey canonical anticommutation relations, *i.e.*,

$$\{b_{\mathbf{n}}^I, b_{\mathbf{n}'}^{\dagger I'}\} \neq \delta_{II'} \delta_{\mathbf{n}\mathbf{n}'}. \quad (2.34)$$

The separation of  $H_{\text{eff}}^{(3)}$  into a canonical kinetic energy and an interaction term is a challenge for the future.

### III. $\sigma$ MODEL REPRESENTATION

Because of the complexity of the third-order effective Hamiltonian, we restrict ourselves henceforth to the second-order theory, in which baryons are a fixed background. The theory defined by  $H_{\text{eff}}^{(2)}$  is a generalized spin model, with spins chosen to be in representations of  $U(N)$  according to the baryon distribution. We review [39] in this section how to convert the spin model into a  $\sigma$  model. The  $\sigma$  field at each site will move in a manifold determined by the baryon number at that site.

#### A. Coherent state basis

We employ a generalization of spin coherent states [43] to derive a path integral for the spin model of  $H_{\text{eff}}^{(2)}$ . We recall that a given site carries generators  $Q_{\mathbf{n}}^\eta$  of  $U(N)$  in a representation with  $N_c$  columns and  $m$  rows, with  $B'_{\mathbf{n}} = m$ .

First we choose a basis for the Lie algebra of  $U(N)$ . This consists of the generators  $S_j^i$ , with  $i, j = 1, \dots, N$ , whose matrix elements in the fundamental rep are

$$(S_j^i)_{fg} = \delta_{if} \delta_{jg}. \quad (3.1)$$

The corresponding charges are

$$\begin{aligned} Q_j^i &= \sum_{\alpha} \psi_{\alpha}^{\dagger} S_j^i \psi_{\alpha} - \frac{1}{2} N_c \delta_j^i \\ &= \sum_{\alpha} \psi_{i\alpha}^{\dagger} \psi_{j\alpha} - \frac{1}{2} N_c \delta_j^i, \end{aligned} \quad (3.2)$$

where we have subtracted a constant for convenience. The Cartan subalgebra consists of the operators

$$H_i = Q_i^i. \quad (3.3)$$

We build the coherent states from the state of highest weight. The highest-weight state  $|\Psi_0\rangle$  in the representation is an eigenstate of the Cartan generators,

$$H_i|\Psi_0\rangle = \begin{cases} (N_c/2)|\Psi_0\rangle & \text{for } i = 1, \dots, m \\ -(N_c/2)|\Psi_0\rangle & \text{for } i = m + 1, \dots, N. \end{cases} \quad (3.4)$$

In this state, the generators take the simple form

$$\langle\Psi_0|Q_j^i|\Psi_0\rangle = \frac{1}{2}N_c\Lambda_{ij}, \quad (3.5)$$

with

$$\Lambda = \begin{pmatrix} \mathbf{1}_m & 0 \\ 0 & -\mathbf{1}_{N-m} \end{pmatrix}. \quad (3.6)$$

The state  $|\Psi_0\rangle$  is invariant (up to a phase) under the subgroup of  $U(N)$  that commutes with  $\Lambda$ ; this is  $U(m) \times U(N - m)$ . The most general rotation of  $|\Psi_0\rangle$  is carried out with the generators  $Q_j^i$  that are *not* in the corresponding subalgebra. We choose parameters  $a_\mu^\lambda$ , with  $\lambda \in [1, m]$  and  $\mu \in [m + 1, N]$ , and write

$$|a\rangle = \exp\left(\sum_{\lambda=1}^m \sum_{\mu=m+1}^N (a_\mu^\lambda Q_\lambda^\mu - a_\mu^{*\lambda} Q_\mu^\lambda)\right)|\Psi_0\rangle. \quad (3.7)$$

The only generators  $Q_\lambda^\mu$  that appear in Eq. (3.7) are those that lower an  $H_i$  that starts from  $N_c/2$  in Eq. (3.4) while raising another  $H_i$  that starts from  $-N_c/2$ . Any other generator would annihilate  $|\Psi_0\rangle$  and thus give no effect in the exponential.

The coherent states are normalized,

$$\langle a|a\rangle = 1, \quad (3.8)$$

and over-complete,

$$\int da |a\rangle\langle a| = 1. \quad (3.9)$$

In Eq. (3.9) the integral is over the coset space  $U(N)/[U(m) \times U(N - m)]$  (see below). Matrix elements of the generators are given by

$$\langle a|Q_j^i|a\rangle = \frac{1}{2}N_c\sigma_{ij}, \quad (3.10)$$

where the matrix  $\sigma_{ij}$  is given by a unitary rotation from  $\Lambda$ ,

$$\sigma = U(a)\Lambda U(a)^\dagger. \quad (3.11)$$

The matrix  $U(a)$  is built out of the  $m \times (N - m)$  matrix  $a_\mu^\lambda$ ,

$$U = \exp\left[\begin{pmatrix} 0 & a \\ -a^\dagger & 0 \end{pmatrix}\right]. \quad (3.12)$$

$\sigma$  is both Hermitian and unitary.

The manifold of matrices  $\sigma$  is the coset space  $U(N)/[U(m) \times U(N-m)]$ , a sub-manifold of  $U(N)$ . This is because for any matrix  $U(a)$ , one can generate an orbit  $U(a)V$  by multiplying with a matrix

$$V = \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}, \quad (3.13)$$

where  $X \in U(m)$  and  $Y \in U(N-m)$ . All matrices in the orbit will give the same matrix  $\sigma$  when inserted into Eq. (3.11), and thus in integrating over the configuration space of  $\sigma$  one must choose only a single representative of each orbit. This set of representatives, the coset space  $U(N)/[U(m) \times U(N-m)]$ , is the quotient space of the non-invariant subgroup  $U(m) \times U(N-m)$ .

The measure over the coset space must be invariant under unitary rotations,

$$|a\rangle \rightarrow R(V)|a\rangle, \quad (3.14)$$

where  $R(V)$  represents the rotation  $V$  in Hilbert space. Equation (3.10) shows that this is a rotation

$$\sigma \rightarrow V\sigma V^\dagger \quad (3.15)$$

and by Eq. (3.11), this means that a measure in  $U$  must be invariant under  $U \rightarrow VU$ . This fixes the measure uniquely to be the Haar measure in  $U(N)$ , and thus one can integrate over the coset space by integrating with respect to  $U$  over  $U(N)$  and using Eq. (3.11).

A representation whose Young diagram has  $N-m$  rows is the conjugate to the representation with  $m$  rows. Its coherent state space can be constructed to look the same, with only a sign difference. To do this we start with the *lowest-weight* state, which satisfies [cf. Eq. (3.4)]

$$H_i|\Psi_0\rangle = \begin{cases} -(N_c/2)|\Psi_0\rangle & \text{for } i = 1, \dots, m \\ (N_c/2)|\Psi_0\rangle & \text{for } i = m+1, \dots, N. \end{cases} \quad (3.16)$$

This introduces a minus sign into Eq. (3.5). The subsequent steps are identical, with only the replacement of Eq. (3.10) by

$$\langle a|Q_j^i|a\rangle = -\frac{1}{2}N_c\sigma_{ij}. \quad (3.17)$$

Here, too,  $\sigma$  is given in terms of  $\Lambda$  and  $U$  by Eq. (3.11).

## B. Partition function

The partition function  $Z = \text{Tr } e^{-\beta H}$  can be written as a path integral by inserting the completeness relation (3.9) at every slice of imaginary time. This gives

$$Z = \int D\sigma(\tau) \exp -S, \quad (3.18)$$

where the action is

$$S = \int_0^\beta d\tau \left[ \frac{1 - \langle a(\tau)|a(\tau + d\tau)\rangle}{d\tau} + H(\sigma(\tau)) \right]. \quad (3.19)$$

The Hamiltonian  $H(\sigma)$  is a transcription of the quantum Hamiltonian to the classical  $\sigma$  matrices. Starting with the quantum operator  $Q_{\mathbf{n}}^\eta$ , we have

$$\begin{aligned} Q_{\mathbf{n}}^\eta &= \psi_{\mathbf{n}}^\dagger M^\eta \psi_{\mathbf{n}} = M_{ij}^\eta \psi_{\mathbf{n}}^\dagger S_j^i \psi_{\mathbf{n}} \\ &= M_{ij}^\eta Q_j^i(\mathbf{n}) + \frac{1}{2} N_c \text{Tr} M^\eta. \end{aligned} \quad (3.20)$$

Expressed in these variables, the quantum Hamiltonian is<sup>7</sup>

$$H_{\text{eff}}^{(2)} = \sum_{\substack{\mathbf{n}, \mu \\ j \neq 0}} J_j Q_{\mathbf{n}}^\eta Q_{\mathbf{n}+j\hat{\mu}}^\eta (s_\eta^\mu)^{j+1}. \quad (3.21)$$

where  $J_j = (2/N_c)K(j)$ . We transcribe this according to Eq. (3.10) to obtain the classical Hamiltonian,

$$H(\sigma) = \left(\frac{N_c}{2}\right)^2 \sum_{\substack{\mathbf{n}, \mu \\ j \neq 0}} J_j \sigma_{\mathbf{n}}^\eta \sigma_{\mathbf{n}+j\hat{\mu}}^\eta (s_\eta^\mu)^{j+1}, \quad (3.22)$$

where

$$\sigma_{\mathbf{n}}^\eta = \text{Tr} M^{\eta T} \sigma_{\mathbf{n}}. \quad (3.23)$$

Recall that each  $\sigma_{\mathbf{n}}$  is an  $N \times N$  matrix ranging over the coset space appropriate to site  $\mathbf{n}$ .

The time-derivative term in  $S$  is a Berry phase [39]. It can be expressed in terms of the matrix  $U$  that determines  $\sigma$  via Eq. (3.11). The result is<sup>8</sup>

$$S = \int_0^\beta d\tau \left[ -\frac{N_c}{2} \sum_{\mathbf{n}} \text{Tr} \Lambda_{\mathbf{n}} U_{\mathbf{n}}^\dagger \partial_\tau U_{\mathbf{n}} + H(\sigma(\tau)) \right]. \quad (3.24)$$

$\Lambda_{\mathbf{n}}$  will vary from site to site if  $m$  does. If one takes the route of Eq. (3.17) for a representation with  $N - m$  rows, then the kinetic term for that site acquires a minus sign (see below).

The number of colors has largely dropped out of the problem, since  $\sigma$  is just an  $N \times N$  unitary matrix field. The explicit factors of  $N_c$  in Eqs. (3.22) and (3.24) invite a semiclassical approximation in the large- $N_c$  limit. This of course neglects the  $N_c$ -dependence of the couplings  $J_j \sim 1/N_c^2$ , but the common scale of the couplings only serves to set an energy scale. The ground state will be independent of this scale, although correlations and the temperature scale will reflect it. We take the point of view that after all  $N_c = 3$ , and we are interested in properties of the effective theory for this value only. The  $N_c \rightarrow \infty$  limit, for fixed couplings  $J_j$ , will be a device for investigating the properties of a generalized effective theory.

#### IV. ZERO DENSITY

The simplest way to set the baryon density to zero is just to choose  $B_{\mathbf{n}} = 0$  on each site, meaning  $m = N/2 = 2N_f$ . It turns out to be just as easy to consider a slightly generalized

<sup>7</sup> The  $B'$  term from Eq. (2.19) indeed disappears. We have dropped an additive constant that is independent of  $B'$ .

<sup>8</sup> This is correct only if  $U$  is of the form given in Eq. (3.12), and in that case  $U$  cannot be integrated over all of  $U(N)$ .

case [17], in which  $B_{\mathbf{n}}$  is chosen to alternate,  $B_{\mathbf{n}} = \pm b$ , on even and odd sublattices. This means to choose a representation with  $m = N/2 + b$  rows on even sites and  $N - m = N/2 - b$  rows on odd sites, which gives a pair of conjugate representations of  $U(N)$ . In view of Eq. (3.17), we can substitute  $\sigma \rightarrow -\sigma$  on the odd sites and thus have identical manifolds on all sites. The Hamiltonian is then

$$H(\sigma) = \left(\frac{N_c}{2}\right)^2 \sum_{\substack{\mathbf{n}\mu \\ j \neq 0}} J_j (-1)^j \sigma_{\mathbf{n}}^\eta \sigma_{\mathbf{n}+j\hat{\mu}}^\eta \left(s_\eta^\mu\right)^{j+1}. \quad (4.1)$$

$m$  is a new parameter in the theory, and we can ask what value of  $m$  gives the lowest energy for the ground state. We will see that  $m = N/2$  is indeed preferred, but only in the next-to-leading order in  $1/N_c$ .

### A. Large- $N_c$ limit

In the large- $N_c$  limit, we seek the classical saddle point<sup>9</sup> of  $S$ . We assume the saddle is at a configuration  $\sigma_{\mathbf{n}}(\tau)$  that is independent of time, and so we drop the time derivative. We begin with the nearest-neighbor Hamiltonian,<sup>10</sup>

$$\begin{aligned} H &= -J \sum \sigma_{\mathbf{n}}^\eta \sigma_{\mathbf{n}+\hat{\mu}}^\eta \\ &= -\frac{J}{2} \sum \text{Tr} \sigma_{\mathbf{n}} \sigma_{\mathbf{n}+\hat{\mu}}. \end{aligned} \quad (4.2)$$

Again, the matrices  $\sigma_{\mathbf{n}}$  are Hermitian and unitary, and the expansion coefficients  $\sigma_{\mathbf{n}}^\eta$  are real and satisfy  $\sum (\sigma^\eta)^2 = N/2$ . The minimum of  $H$  is clearly at a constant field,  $\sigma_{\mathbf{n}} = \sigma_0$ , which can be diagonalized to  $\sigma_{\mathbf{n}} = \Lambda$  by a  $U(N)$  rotation. This is a Néel state in the original variables. The  $U(N)$  symmetry is broken to  $U(m) \times U(N - m)$  and there are  $2m(N - m)$  Goldstone bosons.

The classical energy density (per link) is  $\epsilon_0 = -JN/2$ , independent of  $m$ . Thus at leading order in  $1/N_c$ , the optimal value of  $m$  is undetermined, and any alternating background of baryon number is equally good.

Since it turns out that the  $1/N_c$  corrections select  $m = N/2$ , let us consider the effect of the  $nnn$  term in  $H$  for this case only. The perturbation is

$$H' = J' \sum_{\mathbf{n}\mu} \sigma_{\mathbf{n}}^\eta \sigma_{\mathbf{n}+\hat{\mu}}^\eta s_\eta^\mu. \quad (4.3)$$

It breaks the  $U(N)$  symmetry to  $SU(N_f)_L \times SU(N_f)_R \times U(1)_A \times U(1)_B$ . Assuming that  $J' \ll J$ , we again seek the minimum energy configuration in the form of a constant field; we minimize

$$\epsilon' = \sum_{\mu\eta} \sigma^\eta \sigma^\eta s_\eta^\mu \quad (4.4)$$

among the  $U(N)$ -equivalent  $\sigma = \sigma_0$  states that minimize the nearest-neighbor action. It is not hard to show (see Appendix A) that  $\epsilon'$  is minimized for  $\sigma = \gamma_0 \otimes \mathbf{1}$ . This is a

<sup>9</sup> Note that the kinetic term is pure imaginary.

<sup>10</sup> This classical Hamiltonian is ferromagnetic; the antiferromagnetic nature of the quantum Hamiltonian (3.21) is preserved by the alternating signs in the time-derivative term.

condensate that is symmetric under the vector generators  $M^\eta = \mathbf{1} \otimes \lambda^a$  but not under the axial generators  $\gamma_5 \otimes \lambda^a$ , and thus it breaks the chiral symmetry to the vector subgroup,  $SU(N_f)_V \times U(1)_B$ .

## B. $1/N_c$ corrections

Returning to the nearest-neighbor theory, we consider fluctuations around the  $\sigma_{\mathbf{n}} = \Lambda$  minimum of  $S$ . First we rescale  $\tau \rightarrow 2\tau/N_c$  in order to put the kinetic and potential terms on an equal footing, giving

$$S = \frac{N_c}{2} \int_0^{\bar{\beta}} d\tau \left[ - \sum_{\mathbf{n}} (-1)^n \text{Tr} \Lambda U_{\mathbf{n}}^\dagger \partial_\tau U_{\mathbf{n}} - \frac{J}{2} \sum_{\mathbf{n}\mu} \text{Tr} \sigma_{\mathbf{n}} \sigma_{\mathbf{n}+\hat{\mu}} \right], \quad (4.5)$$

with  $\bar{\beta} = N_c \beta / 2$ . Recalling Eq. (3.12), we can write

$$U_{\mathbf{n}} = e^{A_{\mathbf{n}}}, \quad (4.6)$$

where  $A_{\mathbf{n}}$  is anti-Hermitian and anticommutes with  $\Lambda$ . It is more convenient to work with the Hermitian matrix

$$L_{\mathbf{n}} = 2A_{\mathbf{n}}\Lambda, \quad (4.7)$$

in terms of which we expand

$$\sigma_{\mathbf{n}} = \Lambda + L_{\mathbf{n}} - \frac{1}{2} L_{\mathbf{n}}^2 \Lambda. \quad (4.8)$$

If we further expand  $L_{\mathbf{n}}$  in the basis of generators of  $U(N)$ ,

$$L_{\mathbf{n}} = \sum_{\eta} l^\eta M^\eta, \quad (4.9)$$

we find that the  $l^\eta$  corresponding to generators of  $U(m) \times U(N-m)$  vanish; this is the subgroup under which the vacuum is symmetric. The field  $L_{\mathbf{n}}$  thus contains  $2m(N-m)$  real degrees of freedom, corresponding to the Goldstone bosons.

We expand  $U_{\mathbf{n}}$  and  $\sigma_{\mathbf{n}}$  in powers of  $L_{\mathbf{n}}$ ; using Eq. (4.9), we obtain to second order

$$S = S_0 + \frac{N_c}{2} \int d\tau \sum_{\mathbf{n}} \left[ \frac{(-1)^n}{8} C^{m\eta'} l_{\mathbf{n}}^\eta \partial_\tau l_{\mathbf{n}}^{\eta'} + \frac{J}{8} \sum_{\mu} (l_{\mathbf{n}+\hat{\mu}} - l_{\mathbf{n}})^2 \right]. \quad (4.10)$$

The coefficient matrix is

$$C^{m\eta'} = \text{Tr} (\Lambda [M^\eta, M^{\eta'}]), \quad (4.11)$$

and the classical energy is  $S_0 = -\frac{J}{2} \left(\frac{N_c}{2}\right)^2 N N_s d \beta$ .  $C$  is antisymmetric and purely imaginary; we show in Appendix B that  $C$  has eigenvalues  $\pm 1$ , each with degeneracy  $m(N-m)$ . We change basis so as to diagonalize  $C$ , and write the index  $\eta$  as the compound  $(\alpha, \pm)$  with  $\alpha = 1, \dots, m(N-m)$  and the  $\pm$  corresponding to the eigenvalue of  $C$ . Since the original  $l^\eta$  are real, we have

$$(l^{\alpha+})^* = l^{\alpha-}. \quad (4.12)$$

Thus we eliminate  $l^{\alpha-}$  and write

$$S = S_0 + \frac{N_c}{8} \int d\tau \sum_{\mathbf{n}} \left[ (-1)^{n_i} \text{Im} l_{\mathbf{n}}^{\alpha+*} \partial_\tau l_{\mathbf{n}}^{\alpha+} + \frac{J}{2} \sum_{\mu} |l_{\mathbf{n}+\hat{\mu}}^{\alpha+} - l_{\mathbf{n}}^{\alpha+}|^2 \right]. \quad (4.13)$$



The alternating sign in Eq. (4.13) is what makes the theory antiferromagnetic. It forces us to differentiate between even and odd sites, and we transform to momentum space as follows (dropping the + superscript):

$$l_{\mathbf{n}}^{\alpha} = \sqrt{\frac{2}{N_s}} \sum_{\mathbf{k}} \begin{cases} l_{1,\mathbf{k}}^{\alpha} e^{i\mathbf{k}\cdot\mathbf{n}} & \mathbf{n} \text{ even,} \\ l_{2,\mathbf{k}}^{\alpha} e^{i\mathbf{k}\cdot(\mathbf{n}-\hat{z})} & \mathbf{n} \text{ odd.} \end{cases} \quad (4.14)$$

The even sites comprise an *fcc* lattice with lattice constant 2, and the momenta  $\mathbf{k}$  take values in its Brillouin zone. We obtain

$$S = S_0 + \frac{N_c}{8} \sum_{\mathbf{k}} \int d\tau (l_1^* \quad l_2^*)_{\mathbf{k}}^{\alpha} \mathcal{M}(\mathbf{k}) \begin{pmatrix} l_1 \\ l_2 \end{pmatrix}_{\mathbf{k}}^{\alpha}. \quad (4.15)$$

Here

$$\mathcal{M}(\mathbf{k}) = \begin{pmatrix} Jd - \partial_{\tau} & -Jd\gamma(\mathbf{k}) \\ -Jd\gamma(\mathbf{k}) & Jd + \partial_{\tau} \end{pmatrix} \quad (4.16)$$

and  $\gamma(\mathbf{k}) = \frac{1}{d} \sum_{\mu} \cos k_{\mu}$ .

The gaussian path integral over the  $l$  field now gives the free energy,

$$F = F_0 + m(N - m) \frac{N_c}{2} \sum_{\mathbf{k}} \left[ \frac{1}{\beta} \log \left( 2 \sinh \frac{\bar{\beta}\omega(\mathbf{k})}{2} \right) - \frac{Jd}{2} \right], \quad (4.17)$$

where  $\omega(\mathbf{k}) = dJN_c \sqrt{1 - \gamma^2(\mathbf{k})}$  and  $F_0 = -\frac{1}{2} JdNN_s(N_c/2)^2$ . For the ground state energy, we take  $\beta \rightarrow \infty$  and obtain (restoring all constants)

$$E_0 = -JN_sNd \left( \frac{N_c}{2} \right)^2 \left[ 1 + \frac{1}{N_c} \frac{m(N - m)}{N} \int_{\text{BZ}} \left( \frac{dk}{2\pi} \right)^d \left( 1 - \sqrt{1 - \gamma^2(\mathbf{k})} \right) \right]. \quad (4.18)$$

This is exactly the result of Smit [17]. The  $O(1/N_c)$  corrections lift the degeneracy of the ground states with different values of  $m$ . The integral in Eq. (4.18) is positive and its coefficient contains the number of Goldstone bosons. Thus the state of lowest energy is that with  $m = N/2$ , and the symmetry breaking scheme is  $U(N) \rightarrow U(N/2) \times U(N/2)$ . Further breaking by the *nnn* terms was discussed above.

## V. NONZERO BARYON DENSITY

The zero-density theories considered in the preceding section were defined by selecting representations with  $m$  and  $N - m$  rows on alternating sites. For any  $m$ , this led to a  $\sigma$  model with identical degrees of freedom on all sites—after redefinition of the spins on the odd sublattice—and ferromagnetic couplings.<sup>11</sup> We eventually settled on  $m = N/2$  as the background that gives the ground state of lowest energy.

<sup>11</sup> The  $(-1)^{\mathbf{n}}$  factor in the kinetic energy retained information about the antiferromagnetic nature of the quantum problem; it did not affect the classical analysis.

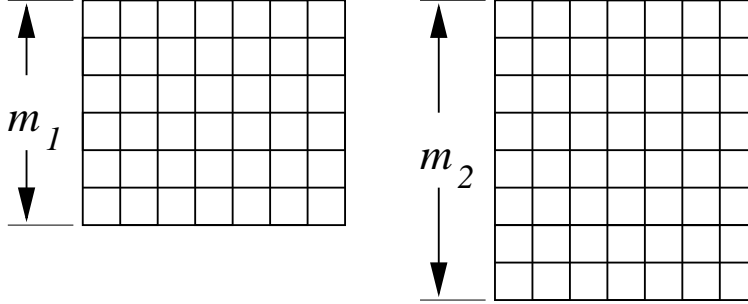


FIG. 3: Two  $U(N)$  spins in different representations

Introducing non-zero baryon density means changing  $m$  on some sites of the lattice. Since in general there will be representations on different sites that are not mutually conjugate, different sites will carry  $\sigma$  variables that do not live in the same submanifold of  $U(N)$ . We here limit ourselves to the simpler case of uniform  $m$ , where adjacent sites carry identical spins—but the coupling is *antiferromagnetic*.

In order to learn how to work with such a theory, we begin by studying the two-site problem. The results of this study will lead directly to an *ansatz* for the ground state of a lattice with a fixed density of baryons.

## A. The two-site problem

### 1. Classical solution

Consider, therefore, two sites with quantum spins  $Q_1$  and  $Q_2$  that carry representations of  $U(N)$  with  $m_1$  and  $m_2$  rows, and  $N_c$  columns (see Fig. 3). The quantum Hamiltonian is

$$H = JQ_1^m Q_2^m, \quad (5.1)$$

an antiferromagnetic coupling. The corresponding classical  $\sigma$  model has the interaction Hamiltonian

$$H(\sigma) = \frac{J}{2} \left( \frac{N_c}{2} \right)^2 \text{Tr} \sigma_1 \sigma_2, \quad (5.2)$$

where

$$\sigma_i = U_i \Lambda_i U_i^\dagger. \quad (5.3)$$

The two  $\Lambda$  matrices reflect the different values of  $m_i$  according to

$$\Lambda_i = \begin{pmatrix} \mathbf{1}_{m_i} & 0 \\ 0 & -\mathbf{1}_{N-m_i} \end{pmatrix}. \quad (5.4)$$

The  $N_c \rightarrow \infty$  limit is the classical limit, in which we seek values of  $\sigma_{1,2}$  that minimize  $H(\sigma)$ . A global  $U(N)$  rotation, *viz.*,

$$\sigma_i \rightarrow V \sigma_i V^\dagger, \quad (5.5)$$

can be used to diagonalize  $\sigma_1$  so that  $\sigma_1 = \Lambda_1$ . Now we have to minimize  $\text{Tr} \Lambda_1 \sigma_2$ . The case of conjugate representations,  $m_2 = N - m_1$ , is easy:  $\sigma_2$  is a unitary rotation of  $\Lambda_2$ , which

(in this case) can be rotated into  $-\Lambda_1$ . This is the unique antiferromagnetic ground state.  $\sigma_1$  and  $\sigma_2$  can be copied to the odd and even sublattices of an infinite lattice to give the classical Néel state considered in the preceding section.

The case  $m_1 = m_2 = m$  is more complex. We consider  $m > N/2$  for definiteness; the other case is similarly handled. We write explicitly<sup>12</sup> [from Eqs. (3.11)–(3.12)]

$$\sigma_2 = \begin{pmatrix} \cos(2\sqrt{aa^\dagger}) & -a \frac{\sin(2\sqrt{a^\dagger a})}{\sqrt{a^\dagger a}} \\ -\frac{\sin(2\sqrt{a^\dagger a})}{\sqrt{aa^\dagger}} a^\dagger & -\cos(2\sqrt{a^\dagger a}) \end{pmatrix}. \quad (5.6)$$

$a^\dagger a$  is a square matrix of dimension  $N - m$  and  $aa^\dagger$  is a square matrix of dimension  $m$ . Since  $\sigma_2$  is a rotation of  $\Lambda$ ,

$$2m - N = \text{Tr } \sigma_2 = \text{Tr } \cos(2\sqrt{aa^\dagger}) - \text{Tr } \cos(2\sqrt{a^\dagger a}), \quad (5.7)$$

and hence the energy is

$$E = \frac{J}{2} \left(\frac{N_c}{2}\right)^2 \text{Tr } \Lambda \sigma \quad (5.8)$$

$$= J \left(\frac{N_c}{2}\right)^2 [\text{Tr } \cos(2\sqrt{a^\dagger a}) + 2m - N]. \quad (5.9)$$

$E$  is minimized when all the eigenvalues of  $a^\dagger a$  are equal to  $\pi^2/4$ . This means that the  $N - m$  column vectors  $\mathbf{a}_i$  form an orthogonal set in  $m$  dimensions, with

$$\mathbf{a}_i^\dagger \cdot \mathbf{a}_j = \left(\frac{\pi}{2}\right)^2 \delta_{ij}. \quad (5.10)$$

Since  $m > N - m$  by assumption, such a set of vectors can always be found.

Since  $a^\dagger a = (\pi^2/4)\mathbf{1}_{N-m}$ , we have  $\sin(2\sqrt{a^\dagger a}) = 0$  and so the off-diagonal blocks of Eq. (5.6) vanish. The lower-right block of  $\sigma_2$  is the unit matrix  $\mathbf{1}_{N-m}$ . We know that  $(\sigma_2)^2 = 1$  since  $\Lambda^2 = 1$  and thus the upper-left block must have eigenvalues  $\pm 1$ . Equating traces of  $\sigma_2$  and  $\Lambda$ , we find that the upper-left  $m \times m$  block must take the form

$$\sigma^{(m)} = U^{(m)} \Lambda^{(m)} U^{(m)\dagger}, \quad (5.11)$$

where

$$\Lambda^{(m)} = \begin{pmatrix} \mathbf{1}_{2m-N} & 0 \\ 0 & -\mathbf{1}_{N-m} \end{pmatrix} \quad (5.12)$$

and  $U^{(m)} \in U(m)$ .  $\sigma^{(m)}$  represents the coset  $U(m)/[U(2m - N) \times U(N - m)]$ .

We conclude that the classical ground state of this  $B \neq 0$  two-site problem is degenerate, even beyond breaking the overall  $U(N)$  symmetry. The solutions can be written as

$$\begin{aligned} \sigma_1 &= \Lambda_1 = \begin{pmatrix} \mathbf{1}_m & 0 \\ 0 & -\mathbf{1}_{N-m} \end{pmatrix}, \\ \sigma_2 &= \begin{pmatrix} \sigma^{(m)} & 0 \\ 0 & \mathbf{1}_{N-m} \end{pmatrix} \end{aligned} \quad (5.13)$$

<sup>12</sup> This generalizes a parametrization found in [44].

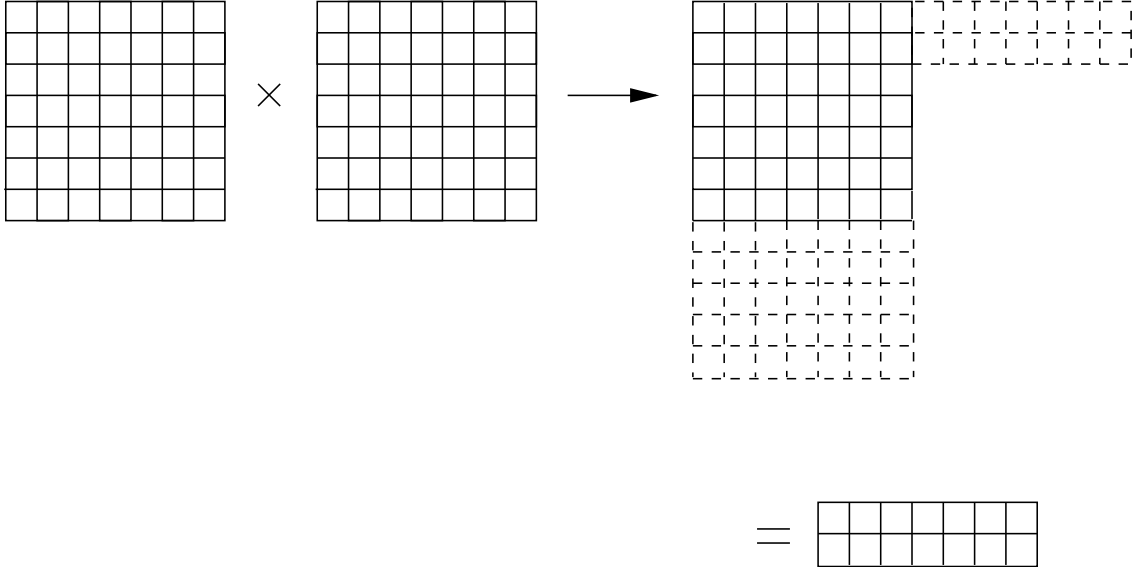


FIG. 4: Coupling two spins in a 7-row representation of  $U(12)$  to the representation with minimal Casimir operator

(to which a global  $U(N)$  rotation can be applied). A particular instance of  $\sigma^{(m)}$  is  $\Lambda^{(m)}$ , given by Eq. (5.12). The symmetry of the vacuum is the set of rotations that leaves both  $\sigma_1$  and  $\sigma_2$  invariant, namely,  $U(2m - N) \times U(N - m) \times U(N - m)$ .

## 2. Quantum fluctuations

The classical solution of the two-site problem will guide us in approaching the problem of an infinite lattice below. We expect that spontaneous symmetry breaking will give a vacuum of the same character, with continuous degeneracy. There are, however, two kinds of degeneracy in the two-site problem: that which results from breaking the global  $U(N)$  to  $U(m) \times U(N - m)$ , and that which comes of breaking the  $U(m)$  subgroup to  $U(2m - N) \times U(N - m)$ . The latter degeneracy is connected with freedom in choosing the orientation of  $\sigma_2$  relative to  $\sigma_1$ . It is instructive to see how quantum mechanical fluctuations lift the classical degeneracies.

The quantum two-site problem is easy to solve. We rewrite the Hamiltonian (5.1) as

$$H = \frac{J}{2} [(Q_1 + Q_2)^2 - Q_1^2 - Q_2^2]. \quad (5.14)$$

$Q_1^2$  and  $Q_2^2$  are constants, the quadratic Casimir operator in the  $m$ -row,  $N_c$ -column representation of  $U(N)$ . The first term in Eq. (5.14) is minimized by coupling  $Q_1$  and  $Q_2$  to the representation that minimizes the Casimir, which is the representation with  $2m - N$  rows and  $N_c$  columns (see Fig. 4). The ground state has discrete degeneracy equal to the dimension of this representation.

The exact quantum solution naturally shows no sign of spontaneous symmetry breaking and hence it is not of much relevance to the infinite volume problem. More interesting is the problem where the state of  $Q_1$  is *fixed* and  $Q_2$  is allowed to vary. This breaks by hand the global  $U(N)$  while allowing quantum fluctuations to lift any remaining degeneracy in the

relative orientation of the two spins, so it can be regarded as quantization in the presence of spontaneous symmetry breaking. In effect, this is mean field theory.

We replace the Hamiltonian (5.1) by

$$H^{MF} = J \sum_{\eta=1}^{N^2} \langle Q_1^\eta \rangle Q_2^\eta. \quad (5.15)$$

To minimize the energy we maximize the mean field by choosing the state of  $Q_1$  to be the highest-weight state. This state diagonalizes the generators  $H_i$  of the Cartan subalgebra while other generators of  $U(N)$  have expectation value zero. Thus

$$H^{MF} = J \sum_{i=1}^N \langle H_{1i} \rangle H_{2i}. \quad (5.16)$$

The operators  $H_i$  all commute, and their eigenvalues make up the weight diagram of the representation.<sup>13</sup>  $H^{MF}$  is a dot product of the weight vectors of the two spins. The energy is minimized by choosing for  $Q_2$  a state that lies opposite the highest-weight state in the weight diagram. As shown in the example of Fig. 5, this still leaves a degeneracy, albeit a discrete one. We stress that this degeneracy comes from freedom in the relative orientation of  $Q_1$  and  $Q_2$ ; it remains after quantum fluctuations lift the continuous degeneracy of the classical system.

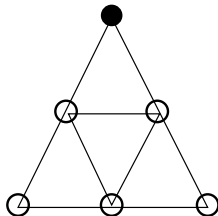


FIG. 5: Weight diagram for  $N = 3$ ,  $m = 1$ ,  $N_c = 2$  [the sextet of  $SU(3)$ ]. The highest-weight state lies at the top of the triangle; all states at the base minimize  $H^{MF}$ . All values of  $N_c$  give the same triangular shape, with multiplicity of one along the boundary. There are  $N_c + 1$  states at the base. For  $N = 4$  the triangle becomes a tetrahedron, and there are  $(N_c + 1)(N_c + 2)/2$  states at its base.

In the  $N_c \rightarrow \infty$  limit, the discrete degeneracy becomes infinite and presumably it is well described by the continuous degeneracy of the classical problem.

## B. Infinite lattice

At  $N_c = \infty$  we seek the saddle point of the action, which we assume to be a time-independent configuration. The classical Hamiltonian of the  $\sigma$  model is

$$H = \frac{J}{2} \sum_{\mathbf{n}, \mu} \text{Tr} \sigma_{\mathbf{n}} \sigma_{\mathbf{n}+\hat{\mu}}. \quad (5.17)$$

<sup>13</sup> More precisely, the weight diagram shows eigenvalues of the  $N - 1$  traceless diagonal generators of  $SU(N)$ . These can be obtained by isolating the  $U(1)$  member of the set  $H_i$  and taking linear combinations of the rest.

Seeking an antiferromagnetic ground state, we set  $\sigma_n = \Lambda$  on the sublattice of even sites. The odd sites are then governed by

$$H^{\text{odd}} = Jd \sum_{\mathbf{n} \text{ odd}} \text{Tr} \Lambda \sigma_{\mathbf{n}}. \quad (5.18)$$

This is just the two-site problem studied above, replicated over the lattice. As we saw above, the ground state configuration is degenerate with respect to the configuration at each odd site,

$$\sigma_{\mathbf{n}} = \begin{pmatrix} \sigma_{\mathbf{n}}^{(m)} & 0 \\ 0 & \mathbf{1}_{N-m} \end{pmatrix} \quad (5.19)$$

A uniform choice for the odd sites,  $\sigma_{\mathbf{n}}^{(m)} = \Lambda^{(m)}$  for instance, breaks the  $U(N)$  symmetry to  $U(2m - N) \times U(N - m) \times U(N - m)$ ; a non-uniform choice can break the symmetry all the way to  $U(N - m)$ . The entropy of this classical ground state is evidently proportional to the volume.

As noted for the two-site problem, the continuous degeneracy of the ground state is an artifact of the classical,  $N_c \rightarrow \infty$  limit. Quantum fluctuations will spread each odd spin's wave function over the  $U(m)/[U(2m - N) \times U(N - m)]$  manifold. A mean-field *ansatz* for the even spins, as noted in the discussion of the two-spin problem, will still leave a discrete degeneracy for the odd spins; the symmetry breaking scheme will depend on how the odd spins are allocated to the available states. Furthermore, one can contemplate making a non-uniform *ansatz* for the *even* spins as well, reducing the phase space available for odd spins with unequal neighbors but adding entropy on the even sublattice. The situation is reminiscent of that in the antiferromagnetic Potts model [45], the phase structure of which is not yet understood.

This Potts-like discrete degeneracy, however, is an artifact of the mean-field approach that, like the classical *ansatz*, assumes a fixed state for the spins on the even sites. An essential difference between our  $\sigma$  model and the Potts model is that our degrees of freedom are continuous and will fluctuate as soon as they are allowed to do so. A given odd spin will *not* be surrounded by a uniform fixed background; the neighboring even spins will be influenced by *all* their odd neighbors, and will induce an interaction among the odd spins that makes them rotate together. This should reduce the entropy of the ground state to zero. The systematic way to see this effect is to carry out a  $1/N_c$  expansion around the classical *ansatz*, which we do in Appendix C. The result is a ferromagnetic interaction among the  $\sigma_{\mathbf{n}}$  on the odd sites. Thus the ground state turns out to be the two-site solution, replicated *uniformly* over the lattice:

$$\begin{aligned} \sigma_{\mathbf{n}} &= \Lambda, & \mathbf{n} \text{ even,} \\ \sigma_{\mathbf{n}} &= \begin{pmatrix} \sigma^{(m)} & 0 \\ 0 & \mathbf{1}_{N-m} \end{pmatrix}, & \mathbf{n} \text{ odd,} \end{aligned} \quad (5.20)$$

where  $\sigma^{(m)} = U^{(m)} \Lambda^{(m)} U^{(m)\dagger}$  is a *global* degree of freedom. The symmetry group of the vacuum is  $U(2m - N) \times U(N - m) \times U(N - m)$ .

## VI. SUMMARY AND DISCUSSION

Let us summarize the results presented in this paper. In the vacuum sector, we have rederived Smit's result for the lowest-energy configuration of alternating  $B_{\mathbf{n}} = \pm(m - N/2)$

sites. The result is indeed  $B_{\mathbf{n}} = 0$ ; the  $U(4N_f)$  symmetry of the nearest-neighbor theory is spontaneously broken to  $U(2N_f) \times U(2N_f)$ . We extended this result to the  $nnn$  theory and found that its  $U(N_f) \times U(N_f)$  chiral symmetry is broken to the vector  $U(N_f)$  flavor subgroup. Adding net baryon number to the system, we examined the case of uniform baryon density,  $B_{\mathbf{n}} = m - 2N_f > 0$ . Here our study was limited to the nearest-neighbor theory, and we found a Néel-like ground state that breaks  $U(N)$  to  $U(2m - N) \times U(N - m) \times U(N - m)$ . The number of Goldstone bosons  $n_{GB}$  thus depends on the baryon density  $B_{\mathbf{n}}$  as

$$n_{GB} = 2(2N_f - |B_{\mathbf{n}}|)(2N_f + 3|B_{\mathbf{n}}|). \quad (6.1)$$

Directions for future work begin with adding  $nnn$  interactions to the  $B > 0$  theory and extracting from it a prediction for the breaking of the continuum-like chiral symmetry. (The axial  $U(1)$  symmetry can be broken by hand.) Another direction is to gain greater freedom in fixing the baryon density. A constant value of  $B_{\mathbf{n}} > 0$  means a baryon density that is close to the maximum allowed on the lattice; the density can be lowered by setting  $B_{\mathbf{n}} \neq 0$  only on a sparse sublattice, along the lines shown in [14]. An ultimate goal, as for the Hubbard model, is the incorporation of the third-order term in the effective Hamiltonian in order to have a theory with dynamical baryons. Perhaps an instructive half-measure would be to study the second-order theory in the presence of a disordered baryon background.

The strong-coupling effective theory can be regarded as a QCD-like model, possessing gauge invariance and the correct degrees of freedom. In that case the lattice spacing is merely a parameter, an overall scale. More insight can be gained by considering the strong-coupling theory to represent QCD at large distances, derived by some renormalization-group transformation from a weak-coupling short-distance Hamiltonian. On the one hand, one would expect any such effective Hamiltonian to contain many terms of great complexity; on the other hand, a simple lattice theory such as ours might offer a qualitative approximation to the real theory (as long as one accepts the loss of Lorentz invariance). We can estimate the lattice spacing to be some scale at which the running QCD coupling is large, certainly greater than the radius of a proton. The limitation that the lattice puts on the density then becomes a physical issue. Taking the lattice spacing  $a$  to be on the order of 1 fm, the highest baryon density allowed by the lattice is  $2N_f \text{ fm}^{-3}$ . For low values of  $N_f$  this may not be enough to see finite-density phase transitions, in particular a transition to color superconductivity. Perhaps a way out is to consider an unphysically large number of flavors.

## Acknowledgments

This work was begun while B.S. was on sabbatical leave at the Center for Theoretical Physics at MIT. He thanks the members of the center for their hospitality. B.B. thanks the organizers of the 19th Jerusalem Winter School in Theoretical Physics for their hospitality. We thank Assa Auerbach, Richard Brower, Krishna Rajagopal, Subir Sachdev, and Uwe-Jens Wiese for helpful discussions. This work was supported by the U.S. Department of Energy under grant no. DF-FC02-94ER40818 and by the Israel Science Foundation under grant no. 222/02-1.

## APPENDIX A: MINIMIZING THE NEXT-NEAREST-NEIGHBOR TERM

The  $s_\eta^\mu$  signs are defined only when the  $\sigma^\eta$  are written in the basis  $M^\eta = \Gamma^A \otimes \lambda^a = \rho^\alpha \otimes \sigma^\beta \otimes \lambda^a$ . We choose a chiral basis for the gamma matrices, so that  $\gamma_5 = \rho^3$ ,  $\alpha_i = \rho^3 \sigma^i$ , and  $\beta = \rho^1$ . The energy (4.4) is a sum of squares,

$$\epsilon' = \sum_\eta A_\eta (\sigma^\eta)^2, \quad (\text{A1})$$

with the constraint  $\sum_\eta (\sigma^\eta)^2 = N/2$ . The coefficients  $A_\eta = \sum_\mu s_\eta^\mu$  take on the values  $\{-3, -1, 1, 3\}$ . The minimum of  $\epsilon'$  occurs when all  $\sigma^\eta$  are zero except those corresponding to  $A_\eta = -3$ , namely, those for  $M^\eta = \beta \otimes \lambda^a = \rho^1 \otimes \lambda^a$  and  $M^\eta = \beta \gamma_5 \otimes \lambda^a = \rho^2 \otimes \lambda^a$ ; the energy is independent of these  $\sigma^\eta$ . Thus the set of solutions can be written in the form

$$\sigma_0 = \begin{pmatrix} 0 & 0 & U & 0 \\ 0 & 0 & 0 & U \\ U^\dagger & 0 & 0 & 0 \\ 0 & U^\dagger & 0 & 0 \end{pmatrix} = \frac{\rho^1 + i\rho^2}{2} \otimes U + \frac{\rho^1 - i\rho^2}{2} \otimes U^\dagger. \quad (\text{A2})$$

Recalling that  $\sigma_0^2 = 1$ , we have  $UU^\dagger = \mathbf{1}$ , so  $U \in U(N_f)$ . A chiral rotation  $\sigma_0 \rightarrow V^\dagger \sigma_0 V$ , with

$$V = \begin{pmatrix} U & 0 & 0 & 0 \\ 0 & U & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & \mathbf{1} \end{pmatrix} = \frac{1}{2}(1 + \rho^3) \otimes U + \frac{1}{2}(1 - \rho^3) \otimes \mathbf{1}, \quad (\text{A3})$$

turns  $\sigma_0$  into  $\rho^1 = \gamma_0$ , which is invariant only under vector transformations generated by  $\mathbf{1} \otimes \lambda^a$ .

## APPENDIX B: THE MATRICES $C$ AND $D_n$

To analyze the matrices  $C$  and  $D_n$ , given by Eqs. (4.11) and (C18), we begin with the  $U(N)$  generators  $M^\eta$  that lie outside the subalgebra  $H = U(m) \times U(N-m)$  that commutes with

$$\Lambda = \begin{pmatrix} \mathbf{1}_m & 0 \\ 0 & -\mathbf{1}_{N-m} \end{pmatrix}. \quad (\text{B1})$$

We choose for them a basis  $M^{pq1}$  and  $M^{pq2}$  with  $p = 1, \dots, m$  and  $q = m+1, \dots, N$ , given by

$$(M^{pq1})_{fg} = \frac{1}{2} (\delta_{pf} \delta_{qg} + \delta_{pg} \delta_{qf}) \quad (\text{B2})$$

$$(M^{pq2})_{fg} = \frac{i}{2} (\delta_{pf} \delta_{qg} - \delta_{pg} \delta_{qf}). \quad (\text{B3})$$

Since the coset space  $U(N)/H$  is a symmetric space, the commutator  $[M^{pqa}, M^{p'q'a'}]$  lies in  $H$ ; in order for  $\text{Tr}(\Lambda[M^{pqa}, M^{p'q'a'}])$  to be nonzero, the commutator must have a nonzero component in the Cartan subalgebra of  $H$ . This is only possible if  $a \neq a'$  and  $(p, q) = (p', q')$ . Thus in this basis  $C$  takes the form

$$C = i \begin{pmatrix} 0 & \mathbf{1}_{m(N-m)} \\ -\mathbf{1}_{m(N-m)} & 0 \end{pmatrix}. \quad (\text{B4})$$



Diagonalizing  $C$  gives eigenvalues  $\pm 1$ . The generators corresponding to the basis that diagonalizes  $C$  are

$$(M^{pq+})_{fg} = (M^{pq1} + iM^{pq2})_{fg} = \delta_{pg}\delta_{qf} \quad (\text{B5})$$

$$(M^{pq-})_{fg} = (M^{pq1} - iM^{pq2})_{fg} = \delta_{pf}\delta_{qg}. \quad (\text{B6})$$

As for  $D$ : The only anticommutators among the  $M^{pq\pm}$  that do not vanish [note the bounds on  $(p, q)$ ] are between  $M^{pq+}$  and  $M^{p'q'-}$ , *viz.*,

$$\{M^{pq+}, M^{p'q'-}\}_{fg} = \delta_{qq'}\delta_{pf}\delta_{p'g} + \delta_{pp'}\delta_{q'f}\delta_{qg}. \quad (\text{B7})$$

Noting that  $M^{pq-} = (M^{pq+})^\dagger$ , we find that  $D_{\mathbf{n}}$  takes the block-diagonal form

$$D_{\mathbf{n}}^{pq\pm, p'q'\pm} = - \sum_{\mathbf{m}(\mathbf{n})} \text{Tr} \left[ \{(M^{pq\pm})^\dagger, M^{p'q'\pm}\} \begin{pmatrix} \sigma_{\mathbf{m}}^{(m)} & 0 \\ 0 & -\mathbf{1}_{N-m} \end{pmatrix} \right] \quad (\text{B8})$$

$$= \delta_{qq'} \sum_{\mathbf{m}(\mathbf{n})} (\mathbf{1}_m - \sigma_{\mathbf{m}}^{(m)})_{pp'}. \quad (\text{B9})$$

We summarize this by writing

$$D = \begin{pmatrix} E_{\mathbf{n}} & 0 \\ 0 & E_{\mathbf{n}} \end{pmatrix} \otimes \mathbf{1}_{N-m}, \quad (\text{B10})$$

where  $E_{\mathbf{n}}$  is an  $m \times m$  matrix given by

$$E_{\mathbf{n}} = \sum_{\mathbf{m}(\mathbf{n})} (\mathbf{1}_m - \sigma_{\mathbf{m}}^{(m)}). \quad (\text{B11})$$

It is easy to prove that the eigenvalues of  $E$  range from 0 to  $4d$ . In particular,  $D$  is positive.

### APPENDIX C: $1/N_c$ CORRECTIONS TO THE $B \neq 0$ PROBLEM

We build on the  $N_c = \infty$  vacua described in Sec. V by allowing fluctuations around them. We let the  $\sigma_{\mathbf{n}}$  on the even sites fluctuate around  $\Lambda$ ; we let the  $\sigma_{\mathbf{n}}$  on odd sites roll freely around the  $U(m)/[U(2m-N) \times U(N-m)]$  manifold covered by Eq. (5.19), and also execute small oscillations off the manifold into the  $U(N)/[U(m) \times U(N-m)]$  coset space. Our goal is an effective action for the classical part (5.19) of the odd spins. To reach this, we integrate out the even spins; the off-manifold fluctuations of the odd spins must be included for consistency in the  $1/N_c$  expansion.

The counterpart of the action (4.5) for our problem has an antiferromagnetic spin-spin interaction, with no  $(-1)^{\mathbf{n}}$  factors. We separate it into odd, even, and coupled terms,

$$\begin{aligned} S &= \frac{N_c}{2} (S^{\text{odd}} + S^{\text{even}} + S^{AF}), \\ S^{\text{odd}} &= \int d\tau \sum_{\mathbf{n} \text{ odd}} \text{Tr} \Lambda U_{\mathbf{n}}^\dagger \partial_\tau U_{\mathbf{n}}, \\ S^{\text{even}} &= \int d\tau \sum_{\mathbf{n} \text{ even}} \text{Tr} \Lambda U_{\mathbf{n}}^\dagger \partial_\tau U_{\mathbf{n}}, \\ S^{AF} &= \int d\tau \sum_{\mathbf{n} \text{ even}} \frac{J}{2} \text{Tr} \sigma_{\mathbf{n}} \sigma_{\mathbf{n}}^o. \end{aligned} \quad (\text{C1})$$

Here  $\sigma_n^o = \sum_{\mathbf{m}(\mathbf{n})} \sigma_{\mathbf{m}}$ , where  $\mathbf{m}(\mathbf{n})$  are the nearest neighbors of the even site  $\mathbf{n}$ . We expand the field on the even sites around  $\sigma_n = \Lambda$  in the manner of Eq. (4.8),

$$\sigma_n = \Lambda + L_n - \frac{1}{2}L^2\Lambda \quad (\mathbf{n} \text{ even}), \quad (\text{C2})$$

while for the odd sites we write (see Appendix D)

$$\sigma_n = U_n \begin{pmatrix} \mathbf{1}_{2m-N} & 0 \\ 0 & -\Lambda' - L' + \frac{1}{2}L'^2\Lambda' \end{pmatrix} U_n^\dagger \quad (\mathbf{n} \text{ odd}), \quad (\text{C3})$$

with

$$\Lambda' = \begin{pmatrix} \mathbf{1}_{N-m} & 0 \\ 0 & -\mathbf{1}_{N-m} \end{pmatrix} \quad (\text{C4})$$

and

$$U_n = \begin{pmatrix} U_n^{(m)} & 0 \\ 0 & \mathbf{1}_{N-m} \end{pmatrix}. \quad (\text{C5})$$

$L_n$  describes the fluctuations of the even spins around their classical value  $\Lambda$ .  $U_n$  rotates the odd spins within the manifold of their classical values, while  $L'_n$  describes their fluctuations outside that manifold. We further define

$$\sigma_n^{\text{cl}} = U_n \begin{pmatrix} \mathbf{1}_{2m-N} & 0 \\ 0 & -\Lambda' \end{pmatrix} U_n^\dagger = \begin{pmatrix} \sigma_n^{(m)} & 0 \\ 0 & \mathbf{1}_{N-m} \end{pmatrix}, \quad (\text{C6})$$

the classical field on the odd sites.

We leave  $S^{\text{odd}}$  alone and expand  $S^{\text{even}}$  and  $S^{AF}$  around the classical values of the fields,

$$S^{\text{even}} = -\frac{1}{4} \int d\tau \sum_{\mathbf{n} \text{ even}} \text{Tr} \Lambda L_n \partial_\tau L_n, \quad (\text{C7})$$

$$S^{AF} = S_0 + \frac{J}{2} \int d\tau \sum_{\mathbf{n} \text{ even}} \left( \text{Tr} L_n \bar{\sigma}_n - \frac{1}{2} \text{Tr} L_n^2 \Lambda \bar{\sigma}_n - \text{Tr} L_n \bar{L}_n \right) \\ + dJ \int d\tau \sum_{\mathbf{n} \text{ odd}} \left( -\text{Tr} \Lambda \tilde{L}_n + \frac{1}{2} \text{Tr} \tilde{L}_n^2 \right). \quad (\text{C8})$$

Here

$$\tilde{L}_n = U_n \begin{pmatrix} 0 & 0 \\ 0 & L'_n \end{pmatrix} U_n^\dagger \quad (\text{C9})$$

is the rotated fluctuation field on the odd sites, and the Hermitian matrices  $\bar{\sigma}_n$  and  $\bar{L}_n$  are sums over the odd neighbors of the even site  $\mathbf{n}$ ,

$$\bar{\sigma}_n = \sum_{\mathbf{m}(\mathbf{n})} \sigma_{\mathbf{m}}^{\text{cl}}, \quad (\text{C10})$$

$$\bar{L}_n = \sum_{\mathbf{m}(\mathbf{n})} \tilde{L}_{\mathbf{m}}. \quad (\text{C11})$$

Since both  $\bar{\sigma}_n$  and  $\Lambda$  are block diagonal, the first trace in each integral in Eq. (C8) is zero.

Now we organize the partition function as follows:

$$Z = \int \left( \prod_{\mathbf{n} \text{ odd}} d\sigma_n \right) \exp \left[ -\frac{N_c}{2} \left( S^{\text{odd}} + S_0 + \frac{dJ}{2} \int d\tau \sum_{\mathbf{n} \text{ odd}} \text{Tr} \tilde{L}_n^2 \right) \right] Z_{\text{even}}, \quad (\text{C12})$$

with

$$Z_{\text{even}} = \int \left( \prod_{\mathbf{n} \text{ even}} dL_{\mathbf{n}} \right) \exp \left[ -\frac{N_c}{2} \int d\tau \right. \\ \left. \times \sum_{\mathbf{n} \text{ even}} \left( -\frac{1}{4} \text{Tr} \Lambda L_{\mathbf{n}} \partial_{\tau} L_{\mathbf{n}} - \frac{J}{4} \text{Tr} L_{\mathbf{n}}^2 \Lambda \bar{\sigma}_{\mathbf{n}} - \frac{J}{2} \text{Tr} L_{\mathbf{n}} \bar{L}_{\mathbf{n}} \right) \right]. \quad (\text{C13})$$

$Z_{\text{even}}$  is a product of decoupled single-site integrals. Again we expand in the group algebra,

$$L_{\mathbf{n}} = l_{\mathbf{n}}^{\eta} M^{\eta}, \quad (\text{C14})$$

where the sum is over the  $2m(N-m)$  generators of  $U(N)$  that are not in  $U(m) \times U(N-m)$ .  $\bar{L}_{\mathbf{n}}$  can be expanded similarly and we obtain the following form for the integral over the even fields:

$$Z_{\text{even}} = \int Dl_{\mathbf{n}} \exp \left[ -\frac{N_c}{2} \int d\tau \sum_{\mathbf{n}} \left( l_{\mathbf{n}}^{\eta} \mathcal{M}_{\mathbf{n}}^{\eta\eta'} l_{\mathbf{n}}^{\eta'} + \frac{J}{4} l_{\mathbf{n}}^{\eta} \bar{l}_{\mathbf{n}}^{\eta} \right) \right], \quad (\text{C15})$$

where

$$\mathcal{M}_{\mathbf{n}}^{\eta\eta'} = -\frac{1}{8} C^{\eta\eta'} \partial_{\tau} + \frac{J}{8} D_{\mathbf{n}}^{\eta\eta'}. \quad (\text{C16})$$

The matrix  $C$  is the same as in Eq. (4.11),

$$C^{\eta\eta'} = \text{Tr} (\Lambda [M^{\eta}, M^{\eta'}]), \quad (\text{C17})$$

while the new matrix  $D$  varies with the site  $\mathbf{n}$  according to the average  $\bar{\sigma}_{\mathbf{n}}$  of its neighboring spins,

$$D_{\mathbf{n}}^{\eta\eta'} = -\text{Tr} (\{M^{\eta}, M^{\eta'}\} \Lambda \bar{\sigma}_{\mathbf{n}}). \quad (\text{C18})$$

We study the two matrices in Appendix B. Diagonalizing  $C$  as before, we arrive at

$$Z_{\text{even}} = \prod_{\mathbf{n}q} \left\{ \int Dl \exp -\frac{N_c}{2} \int d\tau \left[ l^{q\dagger} \hat{\mathcal{M}}_{\mathbf{n}} l^q + \frac{J}{4} \left( l^{q\dagger} \bar{l}_{\mathbf{n}}^q + \bar{l}_{\mathbf{n}}^{q\dagger} l^q \right) \right] \right\}, \quad (\text{C19})$$

where  $\hat{\mathcal{M}}_{\mathbf{n}}$  is the  $m \times m$  matrix

$$\hat{\mathcal{M}}_{\mathbf{n}} = \frac{1}{4} (\partial_{\tau} + J E_{\mathbf{n}}). \quad (\text{C20})$$

The quantities  $l^q$  (and  $\bar{l}^q$ ) for each  $q = 1, \dots, N-m$  are complex  $m$ -component vectors; they are rotations of the  $2m(N-m)$  real components  $l^{\eta}$  (and  $\bar{l}^{\eta}$ ) into the basis that diagonalizes  $C$ . The matrix  $E_{\mathbf{n}}$  is given by Eq. (B11); it carries the dependence on  $\bar{\sigma}_{\mathbf{n}}$ . Performing the gaussian integration we get

$$Z_{\text{even}} = \prod_{\mathbf{n}q} \frac{1}{\text{Det} \hat{\mathcal{M}}_{\mathbf{n}}} \exp \left[ \frac{N_c}{2} \left( \frac{J}{4} \right)^2 \int d\tau \bar{l}_{\mathbf{n}}^{q\dagger} \hat{\mathcal{M}}_{\mathbf{n}}^{-1} \bar{l}_{\mathbf{n}}^q \right]. \quad (\text{C21})$$

Finally we separate the integral (C12) over the odd spins into an integral over the classical field  $\sigma_{\mathbf{n}}^{\text{cl}}$  and an integral over the fluctuations around it. We obtain

$$Z = \int D\sigma_{\mathbf{m}}^{\text{cl}} \exp -\frac{N_c}{2} \left( S_0 + (N-m) \sum_{\mathbf{n}} \text{Tr} \log \hat{\mathcal{M}}_{\mathbf{n}} \right) \\ \times \int D\tilde{l}_{\mathbf{m}} \exp -\frac{N_c}{2} \left\{ S^{\text{odd}} + \int d\tau \left[ \frac{dJ}{2} \sum_{\mathbf{m}q} |\tilde{l}_{\mathbf{m}}^q|^2 - \left( \frac{J}{4} \right)^2 \sum_{\mathbf{n}q} \bar{l}_{\mathbf{n}}^{q\dagger} \hat{\mathcal{M}}_{\mathbf{n}}^{-1} \bar{l}_{\mathbf{n}}^q \right] \right\}. \quad (\text{C22})$$

Here  $\mathbf{m}$  stands for an odd site,  $\mathbf{n}$  for an even one.

Equation (C22) gives an effective action for the classical odd spins  $\sigma_{\mathbf{m}}^{\text{cl}}$ . These enter the exponents through  $\hat{\mathcal{M}}$  [via Eqs. (B11) and (C20)] and through  $\bar{l}_{\mathbf{n}}^q$  [via Eqs. (C9) and (C11)]. The action in the first exponent is minimized when each matrix  $E_{\mathbf{n}}(\sigma_{\mathbf{m}}^{\text{cl}})$  has the largest number of zero eigenvalues, each of which makes  $\text{Tr} \log \hat{\mathcal{M}}_{\mathbf{n}}$  approach  $-\infty$ . It is easy to check that  $E_{\mathbf{n}}$  has  $2m - N$  zero eigenvalues (the maximal number) when the  $\sigma_{\mathbf{m}}^{\text{cl}}$  on *all* the odd sites  $\mathbf{m}(\mathbf{n})$  align, *i.e.*,

$$\sigma_{\mathbf{m}}^{\text{cl}} = \sigma_0 \in U(m)/[U(2m - N) \times U(N - m)]. \quad (\text{C23})$$

Moreover, when Eq. (C23) holds, all the  $\tilde{l}_{\mathbf{m}}^q$ 's align parallel to each other and  $\bar{l}_{\mathbf{n}}^q$  is maximized; also the eigenvalues of  $\hat{\mathcal{M}}_{\mathbf{n}}^{-1}$  are maximized (to  $+\infty$ ). Thus the action in the second exponent also has a minimum at this point in configuration space. These effects add up to an effective *ferromagnetic* interaction among the  $2d$  nearest neighbors  $\mathbf{m}$  of each even site  $\mathbf{n}$ . This effective interaction will align the classical spins on the odd sublattice to the same direction in their submanifold,  $U(m)/[U(2m - N) \times U(N - m)]$ .

The divergences in the effective action have their origin in the fact that the semiclassical corrections are calculated as gaussian integrals in the even fluctuation fields  $L_{\mathbf{n}}$ , and the coefficient matrix  $\mathcal{M}_{\mathbf{n}}$  acquires zero eigenvalues. The correct range of integration over  $L_{\mathbf{n}}$  is of course not infinite, but rather the volume of the  $U(N)/[U(m) \times U(N - m)]$  manifold. This will regulate the divergences, but leave the effective action for the odd spins attractive.

#### APPENDIX D: FLUCTUATIONS ON THE ODD SITES

In the classical analysis, the fields on the odd sites take values in the sub-manifold  $U(m)/[U(2m - N) \times U(N - m)]$  of the manifold  $U(N)/[U(m) \times U(N - m)]$ . We denote these values  $\sigma^{\text{cl}}$ ,

$$\sigma^{\text{cl}} = \begin{pmatrix} \sigma^{(m)} & 0 \\ 0 & \mathbf{1}_{N-m} \end{pmatrix}. \quad (\text{D1})$$

Here

$$\sigma^{(m)} = U^{(m)} \Lambda^{(m)} U^{(m)\dagger}. \quad (\text{D2})$$

with  $U^{(m)} \in U(m)$ , and

$$\Lambda^{(m)} = \begin{pmatrix} \mathbf{1}_{2m-N} & \\ 0 & -\mathbf{1}_{N-m} \end{pmatrix}. \quad (\text{D3})$$

$\sigma^{(m)}$  contains  $2(N - m)(2m - N)$  independent degrees of freedom. Any  $\sigma \in U(N)/[U(m) \times U(N - m)]$  can be written as

$$\sigma = \begin{pmatrix} \cos(2\sqrt{aa^\dagger}) & -a \frac{\sin(2\sqrt{a^\dagger a})}{\sqrt{a^\dagger a}} \\ -\frac{\sin(2\sqrt{a^\dagger a})}{\sqrt{a^\dagger a}} a^\dagger & -\cos(2\sqrt{a^\dagger a}) \end{pmatrix} \quad (\text{D4})$$

[cf. Eq. (5.6)], which coincides with Eq. (D1) if

$$a = U^{(m)} \begin{pmatrix} 0 \\ (\pi/2)\mathbf{1}_{N-m} \end{pmatrix}. \quad (\text{D5})$$

Recall that  $a$  is an  $m \times (N-m)$  matrix, so the zero block has dimensions  $(2m-N) \times (N-m)$ .

We allow motion out of the sub-manifold by allowing  $a$  to vary further,

$$a = U^{(m)} \begin{pmatrix} 0 \\ \bar{a} \end{pmatrix}. \quad (\text{D6})$$

The  $2(N-m)^2$  degrees of freedom in  $\bar{a}$  complement the  $2(N-m)(2m-N)$  degrees of freedom inherent in  $U^{(m)}$  to give  $2m(N-m)$ , the dimensionality of the entire  $U(N)/[U(m) \times U(N-m)]$  coset space.

Writing  $\sigma$  with the generalized  $a$  we have

$$\sigma = U \begin{pmatrix} \mathbf{1}_{2m-N} & 0 & 0 \\ 0 & \cos(2\sqrt{\bar{a}\bar{a}^\dagger}) & -\bar{a} \frac{\sin(2\sqrt{\bar{a}\bar{a}^\dagger})}{\sqrt{\bar{a}\bar{a}^\dagger}} \\ 0 & -\frac{\sin(2\sqrt{\bar{a}\bar{a}^\dagger})}{\sqrt{\bar{a}\bar{a}^\dagger}} \bar{a}^\dagger & -\cos(2\sqrt{\bar{a}\bar{a}^\dagger}) \end{pmatrix} U^\dagger, \quad (\text{D7})$$

with

$$U = \begin{pmatrix} U^{(m)} & 0 \\ 0 & \mathbf{1}_{N-m} \end{pmatrix}. \quad (\text{D8})$$

We can also write this as

$$\sigma = U \begin{pmatrix} \mathbf{1}_{2m-N} & 0 \\ 0 & \sigma^{[2(N-m)]}(\bar{a}, \bar{a}^\dagger) \end{pmatrix} U^\dagger. \quad (\text{D9})$$

$\sigma^{[2(N-m)]}$  is a matrix in the manifold  $U(2(N-m))/[U(N-m) \times U(N-m)]$ . Indeed for  $\bar{a} = (\pi/2)\mathbf{1}_{N-m}$ , we have

$$\sigma^{[2(N-m)]} = \begin{pmatrix} -\mathbf{1}_{N-m} & 0 \\ 0 & \mathbf{1}_{N-m} \end{pmatrix} \equiv -\bar{\Lambda}. \quad (\text{D10})$$

Since  $U(2(N-m))/[U(N-m) \times U(N-m)]$  is a self-conjugate manifold, its structure near  $\sigma^{[2(N-m)]} = -\bar{\Lambda}$  is the same as its structure near  $\sigma^{[2(N-m)]} = \bar{\Lambda}$ , which corresponds to  $\bar{a} = 0$ . Expanding  $\sigma^{[2(N-m)]}$  around  $-\bar{\Lambda}$  gives Eq. (C3).

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