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BUBBLE FREE ENERGY AT THE QUARK–HADRON PHASE TRANSITION

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ABSTRACT. We calculate the free energy of finite droplets of quark–gluon plasma, and of finite hadronic bubbles in the bulk plasma, near the confinement phase transition. We sum over free quark and gluon energy levels in the presence of MIT bag boundary conditions. We find that the curvature term in the free energy, proportional to the radius of the droplet/bubble, is far more important than the contribution of the surface tension, proportional to the radius squared. This affects the critical radius for nucleation of plasma droplets in the superheated hadron gas, and seems to lead to instability of the plasma (even when *not* supercooled) against nucleation of hadron gas bubbles.

I. Introduction

First–order phase transitions generally begin with supercooling (or superheating) followed by the nucleation and growth of bubbles of the equilibrium phase[1]. This kind of dynamics in the transition from quark–gluon plasma to hadron gas should have observable consequences, be it in the early Universe or in relativistic heavy ion collisions. In the former, bubble formation and growth may affect the mass distribution[2−4] and the process of nucleosynthesis[4], and might lead to the formation of strange matter[2]; in the latter, evaporating droplets of plasma may lead to large fluctuations in multiplicity[5] or to structures visible in pion interferometry[6].

Calculation of the nucleation rate[1] of bubbles of the equilibrium phase begins with knowledge of the free energy F of a finite bubble as a function of its radius R. Usually, one assumes that the bubble is very large compared to the correlation length, and one therefore expands the free energy about $R = \infty$,

$$F(R) = \Delta P \cdot \frac{4}{3}\pi R^3 + \sigma \cdot 4\pi R^2 + \cdots , \tag{1.1}$$

where we have retained the terms proportional to the volume and surface area of the bubble, denoting the difference between the pressures of the two phases by ΔP and the surface tension by σ. Terms of higher order in $1/R$ are expected to be insignificant. Both ΔP and σ are functions of temperature, with ΔP vanishing at the transition temperature T_0.

Consider boiling water. When the system is in equilibrium below T_0, both the volume and the surface terms act to suppress the formation of bubbles (see Fig. 1); when the system is superheated above T_0, the volume term encourages growth of bubbles, while the surface term creates an energy barrier. Let us denote by R_c the critical radius, that is, the radius which maximizes $F(R)$ when $T > T_0$. Bubbles are created continually by fluctuations in the free energy. Those which are too small, $R < R_c$, will shrink and vanish. Those which reach R_c will grow until the entire system is converted to the equilibrium phase. Consideration of the fluctuation spectrum in the metastable phase yields the nucleation rate.

At the quark–hadron phase transition, things are somewhat more complicated. The critical radius should be on the order of 1 fm, which is the scale of the strong interactions and also the scale given by the transition temperature. The gluons and light quarks are approximately massless; even if one takes dynamical mass generation into account, their Compton wavelengths will not be much less than 1 fm. Thus the expansion (1.1) of $F(R)$ may not legitimately be truncated after the area term, and the full form of the function should be considered.

We present below a calculation[7] of $F(R)$ in the MIT bag model[8], both for plasma droplets in the hadron gas and for "vacuum bubbles" in the plasma. The plasma is populated with massless gluons, massless u and d quarks, and massive s quarks. Their wave functions obey bag boundary conditions at the interface. The hadron gas is populated with pions, for which only the bulk free energy (the volume term) is taken into account. We reach a surprising conclusion: that the area term in $F(R)$ is insignificant compared to the "curvature term" proportional to R. The reason for this is that neither the light quarks nor the gluons contribute to the surface term, while they do contribute to the other terms. (This fact was noted in Ref. 9.) Thus the surface tension is "accidentally" small.

For plasma droplets in the hadron gas, the volume and curvature terms determine the critical radius and dominate $F(R)$ in its neighborhood. Several other calculations of $F(R)$ have stopped at the area term[10,11]; a calculation which includes the linear term and certain other finite size effects may be found in Ref. 12. Our result should lead to reconsideration of various phenomenological calculations that have used the more limited formulas.

The consequences for hadronic bubbles in the plasma are strange and unexpected. Here the curvature term has the same magnitude but the opposite sign as for the plasma droplets, meaning it acts to make bubbles grow. Hence for $T < T_0$ there is no restoring force on the bubble radius, and hence no barrier—the critical radius is zero. Stranger still, for $T > T_0$ we will have spontaneous nucleation of bubbles which will grow to an equilibrium size $R_{\text{eq}} > 0$ before they are checked by the volume term. Our calculation thus implies that the quark–gluon plasma is unstable against bubble creation, *even above* T_0.

In the next section we calculate the free energy of a spherical plasma droplet in the hadron gas. We solve the Dirac and Maxwell equations for the quark and gluon energy levels, respectively, inside an MIT bag, and then perform the partition sum directly to obtain the free energy as a function of the radius. We demonstrate the strong influence of the massless quarks and gluons on $F(R)$ and list asymptotic approximations for their contributions. We also display graphs of R_c and $F(R_c)$ for superheated plasma droplets.

In section 3 we deal with hadron bubbles in the plasma. We calculate the phase shifts for the quarks and gluons outside the bubble and thence obtain the density of states, whereupon integration yields $F(R)$. After displaying numerical results which show the dominance of a negative curvature coefficient over the positive surface tension, we argue that such behavior is to be expected in view of the results of section 2.

For both the plasma droplet and the hadron bubble, our calculation closely parallels that of Vepstas and Jackson[13] for $T = 0$. For other calculations of zero-point energy in the MIT bag model, see Refs. 14–17. We adopt the point of view[15,17] that the various divergences in the zero-point energy may be eliminated by inclusion of explicit volume, area, curvature, ..., terms in the bag Hamiltonian and suitable renormalization of the couplings; we assume likewise that the finite parts of these couplings at $T = 0$ may be freely adjusted. Since area and curvature terms appear to be unnecessary for fitting the hadron spectrum[9] we assume that these terms are zero at $T = 0$. We take even greater liberty with the bag constant, which we fix at $B^{1/4} = 221$ MeV in order to have a confinement phase transition at $T_0 = 150$ MeV.

Finally, in section 4 we discuss the limitations of our model calculation and how it might be extended and corrected. Even if our physical results do not survive further scrutiny, we expect that more care will be taken in applying bag models to the quark–hadron interface.

II. Free energy of a plasma droplet

We calculate in turn the contributions of the quarks and gluons inside the droplet, and the pions outside. We obtain numerically the quark and gluon energy levels and then perform the partition sum,

$$F(T) = \mp\gamma T \sum_i \log\left(1 \pm e^{-E_i/T}\right) \ . \tag{2.1}$$

In (2.1) the upper sign is for fermions and the lower for bosons; γ is a degeneracy factor. We deal only with the case of zero chemical potential, and choose the strange quark mass to be $m_s = 150$ MeV.

A. Quarks

The eigenvalue equation to be solved for the quark energy E is the time–independent Dirac equation,

$$\left(i\vec{\gamma}\cdot\vec{\nabla} + \gamma^0 E - m\right)\psi = 0 \ , \tag{2.2}$$

along with the MIT bag boundary condition,

$$i\hat{n}\cdot\vec{\gamma}\,\psi = \psi, \qquad r = R, \tag{2.3}$$

where $\hat{n}$ is the inward–pointing unit vector normal to the spherical surface, $\hat{n} = -\hat{r}$. (We suppress the color and flavor indices, and sum over them later.) Upon fixing the quantum numbers j, l, and m according to[18]

$$\psi = \begin{pmatrix} g(r) \\ -if(r)\vec{\sigma}\cdot\hat{r} \end{pmatrix} \phi^l_{jm}(\Omega) \tag{2.4}$$

we obtain the radial equations

$$\frac{dg(r)}{dr} + (1+\kappa)\frac{g(r)}{r} = (E+m)f(r) \tag{2.5}$$

$$\frac{df(r)}{dr} + (1-\kappa)\frac{f(r)}{r} = -(E-m)g(r) \tag{2.6}$$

where for each value of j the parameter κ takes the values

$$\kappa = \pm(j+1/2) \tag{2.7}$$

in accordance with $l = j \pm 1/2$. Combining (2.5) and (2.6) to eliminate $f(r)$ yields the second–order equation

$$\frac{d^2g(r)}{dr^2} + \frac{2}{r}\frac{dg(r)}{dr} - \frac{\kappa(\kappa+1)}{r^2}g(r) + k^2 g(r) = 0 \tag{2.8}$$

where $k^2 \equiv E^2 - m^2$. Eq. (2.8) is solved by a linear combination of spherical Bessel functions,

$$g(r) = Aj_\kappa(kr) + Bn_\kappa(kr) \ , \tag{2.9}$$

whereupon (2.5) gives

$$f(r) = \frac{k}{E+m}[Aj_{\kappa-1}(kr) + Bn_{\kappa-1}(kr)] \ . \tag{2.10}$$

The MIT boundary condition (2.3), along with the decomposition (2.4), gives the condition

$$f(R) = -g(R) \ . \tag{2.11}$$

Since we are solving for wave functions inside the spherical droplet, we demand that the solutions (2.9)–(2.10) be regular at $r = 0$. For $\kappa > 0$, that is, for $l = j + 1/2$, this means setting $B = 0$ to eliminate the n functions. [$\kappa = 0$ is impossible by (2.7).] Thus (2.11) gives the quantization condition

$$j_l(kR) = -\frac{k}{E+m} j_{l-1}(kR) \ , \qquad l = j + 1/2 \ . \tag{2.12}$$

For $\kappa < 0$, or $l = j - 1/2$, we use the relation $n_l(x) = (-)^{l+1} j_{-l-1}(x)$ to make all the indices in (2.9)–(2.10) positive, and only then eliminate the n functions. We then obtain from (2.11)

$$j_l(kR) = \frac{k}{E+m} j_{l+1}(kR) \ , \qquad l = j - 1/2 \ . \tag{2.13}$$

Denoting by E_{njl} the n^{th} solution of (2.12) or (2.13) (according to $l = j \pm 1/2$), the quark free energy is given by

$$F_q(R,T) = -6T \sum_{j=1/2}^{\infty} (2j+1) \sum_{l=j\pm1/2} \sum_n \log\left(1 + e^{-E_{njl}/T}\right) \ , \tag{2.14}$$

where we have fixed the degeneracy factor γ to reflect the presence of 3 colors and of both particle and antiparticle (i.e., positive and negative energy) states. There is one such contribution for each flavor of quark.

When $R \to \infty$, the free energy (2.14) must reduce to

$$F_q(R,T) \to -P_q(T)V \tag{2.15}$$

with

$$P_q(T) = \frac{\gamma_q T}{2\pi^2} \int_0^\infty dk\, k^2 \log\left(1 + e^{\omega(k)/T}\right) \ . \tag{2.16}$$

In this formula $\gamma_q = 2 \cdot 2 \cdot 3 = 12$ for each flavor, and $\omega(k) = \sqrt{k^2 + m_q^2}$.

B. Gluons

We treat the gluon field as eight copies of an Abelian gauge field, and are thus led to the conventional vector multipole expansion[19] for solutions of Maxwell's equations. The separation of variables in spherical coordinates gives for each value of l a transverse-electric and a transverse-magnetic solution,

$$\begin{aligned} \vec{\mathrm{E}}^{TE}_{lm} &= \vec{\nabla} \times \vec{\nabla} \times (\vec{\mathrm{r}}\,\pi_{lm}) \\ \vec{\mathrm{B}}^{TE}_{lm} &= -ik\vec{\nabla} \times (\vec{\mathrm{r}}\,\pi_{lm}) \end{aligned} \tag{2.17}$$

and

$$\begin{aligned} \vec{\mathrm{E}}^{TM}_{lm} &= ik\vec{\nabla} \times (\vec{\mathrm{r}}\,\pi_{lm}) \\ \vec{\mathrm{B}}^{TM}_{lm} &= \vec{\nabla} \times \vec{\nabla} \times (\vec{\mathrm{r}}\,\pi_{lm}) \end{aligned} \tag{2.18}$$

respectively, where

$$\pi_{lm}(r,\Omega) = \sqrt{\frac{2}{\pi}}\,(Cj_l(kr) + Dn_l(kr))\;Y_l^m(\Omega)\;. \tag{2.19}$$

The MIT bag boundary conditions are those of a dual superconductor,

$$\hat{n}\cdot\vec{\mathrm{E}} = \hat{n}\times\vec{\mathrm{B}} = 0 \tag{2.20}$$

and when applied to (2.17) and (2.18) they yield

$$Cj_l(kR) + Dn_l(kR) = 0 \qquad\qquad (TE) \tag{2.21}$$

and

$$\frac{d}{dr}[Crj_l(kr) + Drn_l(kr)]_{r=R} = 0 \qquad\qquad (TM) \tag{2.22}$$

respectively. In solving the problem of gluons inside a spherical bag, we demand the fields be regular at $r = 0$. This forces us to set $D = 0$ in both (2.21) and (2.22), giving us the quantization conditions

$$j_l(kR) = 0 \qquad\qquad (TE) \tag{2.23}$$

and

$$(l+1)j_l(kR) = kRj_{l+1}(kR)\;. \qquad\qquad (TM) \tag{2.24}$$

The gluon contribution to the free energy is a sum over solutions k^a_{nl} of (2.23) and (2.24), where a stands for TE or TM:

$$F_g(R,T) = +8T\sum_{l=1}^{\infty}(2l+1)\sum_{a=TE,TM}\sum_n \log\left(1 - e^{-k^a_{nl}/T}\right)\;, \tag{2.25}$$

where 8 is the color factor for SU(3) gluons.

In the $R\to\infty$ limit, we have $F_g(R,T)\to -P_g(T)V$, with

$$\begin{aligned} P_g(T) &= -\frac{\gamma_g T}{2\pi^2}\int_0^\infty dk\,k^2\log\left(1 - e^{-k/T}\right) \\ &= \gamma_g\frac{\pi^2}{90}T^4\;. \end{aligned} \tag{2.26}$$

The degeneracy factor $\gamma_g = 2\cdot 8 = 16$.

C. Pions and the bag pressure

For the pion gas outside the plasma droplet, we ignore boundary effects. This means that we take into account only the volume term stemming from the excluded volume of the droplet,

$$F_\pi(R,T) = -P_\pi(V_\infty - V_{\text{droplet}}) \ , \tag{2.27}$$

where V_∞ is a fixed large volume and $V_{\text{droplet}} = \frac{4}{3}\pi R^3$. We evaluate P_π as usual,

$$P_\pi(T) = -\frac{\gamma_\pi T}{2\pi^2}\int_0^\infty dk\, k^2 \log\left(1 - e^{-\omega(k)/T}\right) \ , \tag{2.28}$$

where $\omega(k) = \sqrt{k^2 + m_\pi^2}$ and the pion degeneracy factor $\gamma_\pi = 3$. Dropping the R–independent term from (2.27), we are left with

$$F_\pi(R,T) = \frac{4\pi}{3}P_\pi(T)R^3 \ . \tag{2.29}$$

Since we keep only the volume term in the pion free energy, it affects only the transition temperature T_0 [see (2.31) below] and not $F(R)$ at T_0. While one would indeed expect the pions to contribute to the various surface terms as well, their effect on the overall picture should be small because the quarks and gluons have so many more degrees of freedom.

The final ingredient is the confining bag pressure B acting on the outside of the droplet, giving

$$F_{\text{bag}} = \frac{4\pi}{3}BR^3 \ . \tag{2.30}$$

We relate B to the transition temperature T_0 by balancing the pressures of the two phases:

$$P_{ud}(T_0) + P_s(T_0) + P_g(T_0) = P_\pi(T_0) + B \ , \tag{2.31}$$

where $P_{ud} = 2P_q(m=0)$ comes from the light quarks and $P_s = P_q(m = m_s)$ comes from strange quarks. B is a phenomenological parameter, not necessarily the same as the bag constant used in hadron spectroscopy. We henceforth assume a transition temperature $T_0 = 150$ MeV, which implies via (2.31) that $B = (221 \text{ MeV})^4 = 313 \text{ MeV/fm}^3$.

D. Results

We show in Fig. 2 the free energy of a droplet of plasma

$$F(R,T) = F_{ud}(R,T) + F_s(R,T) + F_g(R,T) + (B + P_\pi(T))\frac{4\pi}{3}R^3 \qquad (2.32)$$

for temperatures above, at, and below T_0. The curves are qualitatively similar to those in Fig. 1, but with important differences.

Let us examine the asymptotic (large R) forms of the various contributions to $F(R,T)$. Asymptotic expressions for the density of states of quarks and gluons in the MIT bag have appeared in the literature. For massless quarks,[12]

$$\rho_{ud}^{\mathrm{as}}(k,R) = \frac{k^2 V}{2\pi^2} - \frac{1}{3\pi}R \ ; \qquad (2.33)$$

for massive quarks,[20]

$$\rho_s^{\mathrm{as}}(k,R) = \frac{k^2 V}{2\pi^2} - \frac{k}{8\pi}\left(1 - \frac{2}{\pi}\tan^{-1}\frac{k}{m_s}\right) S \ ; \qquad (2.34)$$

and for gluons,[21]

$$\rho_g^{\mathrm{as}}(k,R) = \frac{k^2 V}{2\pi^2} - \frac{4}{3\pi}R \ . \qquad (2.35)$$

Here V is the volume of the droplet, S its surface area, and R its radius. The free energy of each species for large R is then given by

$$F_i^{\mathrm{as}}(R,T) = \mp\gamma_i T \int_0^\infty dk\, \rho_i^{\mathrm{as}}(k,R) \log\left(1 \pm e^{-\omega(k)/T}\right) \ , \qquad (2.36)$$

yielding the formulas

$$F_{ud}^{\mathrm{as}}(R,T) = \gamma_{ud}\left(-\frac{7\pi^2}{720}T^4 V + \frac{\pi}{36}T^2 R\right) \qquad (2.37)$$

$$F_g^{\mathrm{as}}(R,T) = \gamma_g\left(-\frac{\pi^2}{90}T^4 V + \frac{2\pi}{9}T^2 R\right) \ . \qquad (2.38)$$

The volume term of $F_s^{\mathrm{as}}(R,T)$ is the same as that in (2.37); the area term can be obtained numerically. Note that the massless particles do *not* contribute area terms to the free energy. The massive quarks of course contribute a curvature term as well, but we have not found an analytic expression for the corresponding term in (2.34).

While the asymptotic expressions serve as checks on the numerical calculation, they also provide a framework for description of the final results. Consider Fig. 3, which shows the contributions of the various species to $F(R,T)$ at $T = T_0$, with the volume terms (which cancel against the external pressure at T_0) subtracted. The gluons are responsible for half the free energy for radii as large as 6 fm; likewise, the light quarks dominate the quark contribution as far as $R = 2$ fm. Asymptotically, the massless particles contribute only to the linear term, and thus the strange quarks grow in importance for larger radii because of their R^2 dependence. Still, the massless particles largely dictate the behavior of $F(R,T)$ in the range shown. Thus we conclude that the curvature term dominates the surface piece of the droplet free energy for these radii. Analysis of nucleation based on (1.1) is therefore not justified.

Finally, we present in Fig. 4 plots of the critical radius R_c and the free energy $F(R_c)$ as functions of the degree of superheating $T - T_0$. We display curves for various choices of T_0, adjusted by varying B. Were m_s to be zero, the only dimensionful parameter in the theory would be B or, equivalently, T_0. Then we would have

$$R_c(T) = T_0^{-1} f\left(\frac{T - T_0}{T_0}\right) \tag{2.39}$$

and

$$F(R_c) = T_0 f\left(\frac{T - T_0}{T_0}\right) \ . \tag{2.40}$$

This scaling is broken by the strange quark mass.

III. Free energy of a vacuum bubble

The "vacuum bubble" is a spherical surface of radius R with quarks and gluons outside it obeying bag boundary conditions. Inside the bubble we have the bag pressure B along with a gas of pions (for which we again take into account only the bulk pressure term). The quark and gluon wave functions experience phase shifts $\delta_i(k)$, where i denotes species, angular momentum, etc. These phase shifts change the density of states according to

$$\Delta\rho_i(k) = \frac{1}{\pi}\frac{d\delta_i(k)}{dk} \ . \tag{3.1}$$

For each i there is a contribution to the free energy of the form

$$F_i(R,T) = \mp T \int_0^\infty dk\, \Delta\rho_i(k) \log\left(1 \pm e^{-\omega(k)/T}\right) \ . \tag{3.2}$$

Evidently the phase shifts and $\Delta\rho_i$ are zero in the absence of the bubble, so that $\sum_i F_i(R,T)$ is just the exterior piece of the free energy of the bubble.

It is convenient to integrate (3.2) by parts to obtain

$$F_i(R,T) = -\frac{1}{\pi} \int_0^\infty dk\, \frac{k}{\omega}\, \delta_i(k)\, \frac{1}{e^{\omega/t} \pm 1} \ . \tag{3.3}$$

Again, we deal with the quarks and the gluons in turn.

A. Quarks

Once more we consider solutions of the Dirac equation (2.2) with the bag boundary condition (2.3). This time, since we are *outside* the bag boundary, we have $\hat{n} = \hat{r}$. The new boundary condition replaces (2.11) with

$$f(R) = g(R) \ . \tag{3.4}$$

Given solutions to the radial equations of the form (2.9)–(2.10), we have

$$A\left[j_\kappa(kR) - \frac{k}{E+m} j_{\kappa-1}(kR)\right] = -B\left[n_\kappa(kR) - \frac{k}{E+m} n_{\kappa-1}(kR)\right] \ . \tag{3.5}$$

The ratio A/B gives the phase shift of the wave function at infinity. For the case $\kappa = +(j+1/2) = l$, (2.9) may be approximated for large r as

$$g(r) \sim \frac{1}{kr}\left[A\sin\left(kr - l\frac{\pi}{2}\right) - B\cos\left(kr - \frac{l\pi}{2}\right)\right] \tag{3.6}$$

$$= \frac{1}{kr} C \sin\left(kr - \frac{l\pi}{2} + \delta_j^l\right) \ , \tag{3.7}$$

where (3.7) defines the phase shift δ_j^l. Thus we identify

$$\delta_j^{l=j+1/2} = \tan^{-1}\left(-\frac{B}{A}\right) \ . \tag{3.8}$$

When $\kappa = -(j+1/2) = -l-1$, we replace the Bessel functions in (2.9) by functions with positive index before taking the large r limit as in (3.6). The result is

$$\delta_j^{l=j-1/2} = \tan^{-1}\left(+\frac{A}{B}\right) \ . \tag{3.9}$$

Combining (3.5) with (3.8)–(3.9) gives

$$\delta_j^{l=j\mp1/2} = \tan^{-1}\left[\frac{j_l(kR) \pm \frac{k}{E+m} j_{l\pm1}(kR)}{n_l(kR) \pm \frac{k}{E+m} n_{l\pm1}(kR)}\right] \ . \tag{3.10}$$

Each species of quark thus contributes

$$F_q(R,T) = -\frac{6}{\pi}\sum_{j=1/2}^{\infty}(2j+1)\sum_{l=j\pm1/2}\int_0^\infty dk\,\frac{k}{\omega}\,\delta_j^l(k)\,\frac{1}{e^{\omega/t}+1} \tag{3.11}$$

to the free energy of the bubble.

B. Gluons

For the gluon modes, we have the boundary conditions (2.21)–(2.22). A procedure similar to that used for the quarks gives the phase shifts

$$\delta_l^{TE} = \tan^{-1}\left[\frac{\frac{d}{dr}(rj_l(kr))}{\frac{d}{dr}(rn_l(kr))}\right]_{r=R} \tag{3.12}$$

and

$$\delta_l^{TM} = \tan^{-1}\left[\frac{j_l(kr)}{n_l(kr)}\right] \ . \tag{3.13}$$

The gluons' contribution to the free energy is

$$F_g(R,T) = -\frac{8}{\pi}\sum_{l=0}^{\infty}(2l+1)\sum_{a=TE,TM}\int_0^\infty dk\,\frac{k}{\omega}\,\delta_l^a(k)\,\frac{1}{e^{\omega/t}-1} \ . \tag{3.14}$$

C. Results

Corresponding to (2.32) for the plasma droplet, we have for the vacuum bubble

$$F(R,T) = F_{ud}(R,T) + F_s(R,T) + F_g(R,T) - (B + P_\pi(T))\frac{4\pi}{3}R^3 \ , \tag{3.15}$$

where the sign of the last term reflects the fact that the pion gas is *inside* the bubble, pushing it to expand. We plot $F(R,T)$ in Fig. 5 for temperatures near T_0, and we see immediately that the general forms expected from Fig. 1 are *not* reproduced. The outstanding feature of the curves in Fig. 5 is that their slopes near $R = 0$ are negative. This reflects the facts that (i) the area coefficient, while positive, is very small, and (ii) the curvature coefficient is negative and gives a large term proportional to $-R$. We shall argue shortly that these properties are consistent with the results of section 2. First, however, let us discuss the implications for the physics of the phase transition.

Consider nucleation of a vacuum bubble in a supercooled plasma, $T < T_0$ (solid curve in Fig. 5). The critical radius R_c is zero, and so is the free energy $F(R_c)$ necessary for creation of a growing bubble. This means that supercooling of the plasma is impossible—the hadronic phase begins to appear immediately as T descends past T_0.

Above T_0, it is clear from the dash–dotted curve in Fig. 5 that the plasma is *still* unstable against the formation of bubbles. The system can lower its free energy by nucleating bubbles and letting them grow to the radius $R_{\rm eq}$ which minimizes $F(R)$. This picture persists even for temperatures well above T_0. As can be seen in Fig. 6, both the size and free energy of "equilibrium" bubbles appear to approach finite limits as T grows.

The situation at T_0 (dashed curve) is similar to that above T_0. Here, however, there is no volume term to drive $F(R)$ to ∞ as $R \to \infty$. (This is precisely the condition fixing T_0.) Still, the leading term, the area term, has a positive coefficient, so the curve must turn upwards for sufficiently large R. We can thus lump this case with the $T > T_0$ case. The dashed curve in Fig. 5 shows how small the area coefficient is. (As $T \to T_0^+$ in Fig. 6, both $R_{\rm eq}$ and $F(R_{\rm eq})$ approach large but finite limits corresponding to the minimum of the dashed curve in Fig. 5.)

To connect these results to those of section 2, consider a formal expansion of $F(R)$ about $R = \infty$. Such an expansion might be a result of using the multiple reflection expansion[22,23] to calculate the density of states,

$$\rho(k) = \frac{V}{2\pi^2}k^2 + c_1 k \int dS + c_2 \int dS \left(\frac{1}{R_1} + \frac{1}{R_2}\right) + \cdots . \tag{3.16}$$

The expansion (3.16) applies to a surface of any shape. V is the enclosed volume; the integral in the second term is the surface area; the integrand in the third term is the extrinsic curvature, expressed in terms of the principal radii of curvature; and c_1 and c_2 depend on the type of field and on the boundary conditions, and are functions of k/m. For the interior of a sphere we have

$$\rho(k) = \frac{2}{3\pi}R^3k^2 + c_1\, 4\pi R^2 k + c_2\, 8\pi R + \cdots , \tag{3.17}$$

and (3.16) is evidently an expansion in powers of $(kR)^{-1}$. [Equations (2.33)–(2.35) provide examples.] Inserting $\rho(k)$ into an integral of the form (2.36) gives $F(R)$ as an expansion in R^{-1}. Eq. (3.16), incidentally, explains why the term proportional to R is called the curvature term.

For the case of the plasma droplet, the sign of each term in (3.17) may be inferred from the results of section 2. The volume term is of course positive. The area term, which reflects the surface tension due to the strange quarks, is positive. The curvature term, due largely to the massless particles, is also positive.

Now compare the vacuum bubble. The surface is the same as that of the droplet, but it is viewed from the outside. This means that we must change the sign of R in (3.17). In the first place, the volume term changes sign. The meaning of this is obvious: The fields that were confined in $V_{\rm droplet}$ before are now to be found in $V_\infty - V_{\rm bubble}$, and $\frac{4}{3}\pi R^3$ represents an excluded volume.

When we change the sign of R, the area term remains the same, reflecting a positive surface tension, irrespective of which phase is on which side of the boundary. The curvature term, however, *changes sign*, which is consistent with the fact that the extrinsic curvature is a signed quantity.

To summarize: If we expand

$$F(R) = \Delta P \cdot \frac{4}{3}\pi R^3 + \sigma \cdot 4\pi R^2 + \alpha \cdot 8\pi R + \cdots, \tag{3.18}$$

while *defining* R to be positive, ΔP changes sign at the phase transition as appropriate; σ, the surface tension, is always positive; and α, the curvature coefficient, is positive for the droplet but negative for the bubble. In fact, according to the multiple reflection expansion σ and $|\alpha|$ should be the same for the two cases at any given temperature. Our numerical results show this to be the case.[24]

For sufficiently small R, a negative linear term in (3.18) should lead to a negative slope in $F(R)$, unless terms of higher order in $1/R$ contribute strongly as well. Our numerical results for the vacuum bubble show that a regime with negative slope indeed exists.

IV. Discussion

To say that our results are model-dependent is an understatement. Nevertheless, it is amusing to contemplate their implications for a picture of the quark-gluon plasma. The ordinary, weakly-coupled plasma is apparently unstable against the nucleation of vacuum bubbles. One can always lower the free energy of the plasma by inserting a bubble of radius $R_{\rm eq}$; presumably the state of lowest overall free energy possesses a certain density of such bubbles, where the density is fixed by the bubble–bubble interaction which we have not calculated.

This does not imply that the plasma phase doesn't exist, but rather that it possesses a complex structure involving the admixture of vacuum bubbles. Certainly it is different from the low-temperature phase, which is a gas of pions with no sign of plasma. Consider the dynamics of the "hadronization" phase transition. One can imagine starting with a high-temperature plasma, with its population of bubbles of radius $R_{\rm eq}(T)$. As one cools the system, $R_{\rm eq}$ grows. Bubbles will grow, meet, and coalesce, causing reheating, but equilibrium at any temperature will consist of a fluid of bubbles. Once T_0 is passed, however, the bubbles will grow without restraint until the plasma disappears.

Going in the other direction, production of the plasma by heating the hadron gas will proceed as follows. For $T < T_0$, any plasma droplets which form will vanish quickly. Above T_0, droplets nucleated with $R > R_c$ (see Fig. 4) will grow, meet, and coalesce, leading eventually to the inverted picture, a bulk plasma with shrinking vacuum bubbles. Unlike a conventional phase transition, however, the bubbles will not shrink away to nothing, but will stop shrinking when they reach $R_{\rm eq}$, leaving us with the inhomogeneous phase described above.

Many things are plainly missing in our calculation. Just as interactions among bubbles pose an interesting question, so do interactions inside the bubbles. After all, these "vacuum bubbles" are full of pions. At $T_0 = 150$ MeV, a free pion gas has 0.15 pions/fm^3, which, with a pion radius of 0.6 fm, gives $\frac{4}{3}\pi r_\pi^3 \rho_\pi = 0.14$. This is not a very dilute system, and ideal gas considerations may be inadequate.

The surface energy of this pion gas merits attention. If we were simply to give the pions some definite linear boundary condition at the bag surface, they would produce a surface tension, a curvature term, etc. Recall, however, that the pions comprise only three degrees of freedom, and thus their contribution is perhaps negligible next to that of the quarks and gluons.

A more consistent way to include pion effects, in any case, is via a chiral bag model.[25] The chiral bag boundary condition

$$i\hat{n} \cdot \vec{\gamma}\,\psi = e^{i\tau^a \pi^a \gamma_5}\psi, \qquad r = R, \tag{4.1}$$

couples the quark fields on one side of the boundary with the pion field on the other side in a highly non-linear manner. It may be possible to calculate effects of this coupling in perturbation theory in the pion field.[26] Even so, the gluon boundary condition is unchanged, and the gluon contribution to the surface energy will still consist of a large curvature term and no surface tension.

As noted above, the absence of an area term is characteristic of massless quarks and gluons. Just as the strange quarks contribute a surface tension, so may the light quarks if their thermal masses are taken into account. The question here is, first, how to do this consistently, and, second, what value to take for this mass near the phase transition. The same applies to gluons and their dynamical plasmon mass.

Can our result be changed by adjusting the parameters of the bag model? One might be tempted to add a temperature-independent curvature term to cancel the finite-temperature term near T_0. Not only is this idea unnatural, but it leads to disaster for the hadronic phase. Write the curvature coefficient as $\alpha = \alpha_0 + \alpha_1(T)$, where α_1 is the result of our calculation and α_0 is independent of temperature. In order to stabilize the plasma against vacuum bubbles, the term proportional to α_0 must be positive for bubbles, and hence negative for plasma droplets. As one lowers the temperature into the hadronic phase, $\alpha_1(T)$ disappears, leaving a negative curvature term which will spontaneously nucleate plasma droplets in the vacuum. One thus trades instability of the plasma for instability of the low-temperature vacuum.

Apparently, it is not very easy to make the effect go away in the context of the bag model. One must consider the possibility that the bag is just not a very good model for the quark–hadron interface. After all, is it reasonable to demand that all the dynamics of the interface be dictated by the boundary conditions of the fields on either side? A more realistic picture is offered by soliton bag models[27], where surface tension is due to the variation of an effective scalar field as it interpolates between the free-quark interior of the bag and the quark-excluding exterior. This seems to offer the most promising arena for future attempts to address this problem. Perhaps lattice methods for calculating the interface free energy[28] may also be applied here.

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FIGURE CAPTIONS

FIGURE 1. Schematic representation of bubble free energy in boiling water, retaining only volume and area terms, for $T < T_0$ (solid) and $T > T_0$ (dashed).

FIGURE 2. Free energy of a plasma droplet in the hadron gas for temperatures near $T_0 = 150$ MeV. Top to bottom: $T = 147$ MeV, 150 MeV ($= T_0$), 152 MeV, 154 MeV.

FIGURE 3. Surface free energy at $T = T_0 = 150$ MeV, with volume term subtracted. Bottom to top: strange quark contribution, total quark contribution, total of quarks and gluons.

FIGURE 4. (a) Critical radius R_c and (b) free energy $F(R_c)$ for plasma droplets in a superheated hadron gas, as functions of the degree of superheating. Curves are for choices of $T_0 = 125$, 175, 225 MeV. T_0 increases from top curve to bottom curve in (a), and from bottom curve to top curve in (b).

FIGURE 5. Free energy of a hadron gas bubble in the plasma for temperatures near $T_0 = 150$ MeV. Top to bottom: $T = 155$ MeV, 150 MeV ($= T_0$), 147 MeV.

FIGURE 6. (a) Radius $R_{\rm eq}$ and (b) free energy $F(R_{\rm eq})$ of stable hadron gas bubbles in the plasma for $T > T_0 = 150$ MeV.