Attachment to

T. DeGrand, Y. Shamir, and B. Svetitsky, *SU(4) lattice gauge theory with decuplet fermions:...*, Appendix A

Coefficients for U(*n*) projection (and its derivative): U(3) and U(4)

Set n = 3 or 4 for SU(n)

In[1]:=
```
n = 4;
```

$r_i = \sqrt{g_i}$ -- use instead of g_i everywhere

In[2]:=
```
SG = Table[r_i, {i, 0, n - 1}]
```

Out[2]= $\{r_0, r_1, r_2, r_3\}$

set of symmetric polynomials -- 3 or 4 of them

In[3]:=
```
SP = If[n == 3, {u, v, w}, {u, v, w, x}]
```

Out[3]= $\{u, v, w, x\}$

Vandermonde matrix $\partial c_i / \partial g_j$

In[4]:=
```
VM = Table[r_i^(2 j), {j, 0, n - 1}, {i, 0, n - 1}]; MatrixForm[VM]
```

Out[4]//MatrixForm=

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ r_0^2 & r_1^2 & r_2^2 & r_3^2 \\ r_0^4 & r_1^4 & r_2^4 & r_3^4 \\ r_0^6 & r_1^6 & r_2^6 & r_3^6 \end{pmatrix}$$

Invert it.

In[5]:=
```
VMI = Together[Inverse[VM]]; MatrixForm[VMI]
```

Out[5]//MatrixForm=

$$\begin{pmatrix} \frac{r_1^2 r_2^2 r_3^2}{(-r_0^2+r_1^2)(-r_0^2+r_2^2)(-r_0^2+r_3^2)} & \frac{-r_1^2 r_2^2 - r_1^2 r_3^2 - r_2^2 r_3^2}{(-r_0^2+r_1^2)(-r_0^2+r_2^2)(-r_0^2+r_3^2)} & \frac{r_1^2+r_2^2+r_3^2}{(-r_0^2+r_1^2)(-r_0^2+r_2^2)(-r_0^2+r_3^2)} & -\frac{1}{(-r_0^2+r_1^2)(-r_0^2+r_2^2)(-r_0^2+r_3^2)} \\ \frac{r_0^2 r_2^2 r_3^2}{(r_0^2-r_1^2)(-r_1^2+r_2^2)(-r_1^2+r_3^2)} & \frac{-r_0^2 r_2^2 - r_0^2 r_3^2 - r_2^2 r_3^2}{(r_0^2-r_1^2)(-r_1^2+r_2^2)(-r_1^2+r_3^2)} & \frac{r_0^2+r_2^2+r_3^2}{(r_0^2-r_1^2)(-r_1^2+r_2^2)(-r_1^2+r_3^2)} & -\frac{1}{(r_0^2-r_1^2)(-r_1^2+r_2^2)(-r_1^2+r_3^2)} \\ \frac{r_0^2 r_1^2 r_3^2}{(r_0^2-r_2^2)(r_1^2-r_2^2)(-r_2^2+r_3^2)} & \frac{-r_0^2 r_1^2 - r_0^2 r_3^2 - r_1^2 r_3^2}{(r_0^2-r_2^2)(r_1^2-r_2^2)(-r_2^2+r_3^2)} & \frac{r_0^2+r_1^2+r_3^2}{(r_0^2-r_2^2)(r_1^2-r_2^2)(-r_2^2+r_3^2)} & -\frac{1}{(r_0^2-r_2^2)(r_1^2-r_2^2)(-r_2^2+r_3^2)} \\ \frac{r_0^2 r_1^2 r_2^2}{(r_0^2-r_3^2)(r_1^2-r_3^2)(r_2^2-r_3^2)} & \frac{-r_0^2 r_1^2 - r_0^2 r_2^2 - r_1^2 r_2^2}{(r_0^2-r_3^2)(r_1^2-r_3^2)(r_2^2-r_3^2)} & \frac{r_0^2+r_1^2+r_2^2}{(r_0^2-r_3^2)(r_1^2-r_3^2)(r_2^2-r_3^2)} & -\frac{1}{(r_0^2-r_3^2)(r_1^2-r_3^2)(r_2^2-r_3^2)} \end{pmatrix}$$

Matrix for Eq. (A4)

In[6]:=
```
M = Transpose[VM]; MatrixForm[M]
```

Out[6]//MatrixForm=

$$\begin{pmatrix} 1 & r_0^2 & r_0^4 & r_0^6 \\ 1 & r_1^2 & r_1^4 & r_1^6 \\ 1 & r_2^2 & r_2^4 & r_2^6 \\ 1 & r_3^2 & r_3^4 & r_3^6 \end{pmatrix}$$

RHS of Eq. (A4)

In[7]:=
```
W = Table[1 / r_i, {i, 0, n - 1}]
```

Out[7]= $\left\{\frac{1}{r_0}, \frac{1}{r_1}, \frac{1}{r_2}, \frac{1}{r_3}\right\}$

f_i, from solving Eq. (A4)

In[8]:=
```
F = Together[Inverse[M].W];
```

Write f_i in terms of the symmetric polynomials. (Use **CForm[]** to get output that is usable in a C program.)

In[9]:=
```
For[i = 1, i ≤ n, i++,
  NUM[i] = Simplify[SymmetricReduction[Numerator[Together[F[[i]]]], SG, SP]];
  Print[NUM[i][[1]]]
 ]
```

$-w^3 - u^2 w x + v w x + u \left(v w^2 - v^2 x + x^2\right)$

$2 u^2 v w - u \left(v^3 + 2 w^2\right) - u^3 x + w \left(v^2 + x\right)$

$u^3 v - u^2 w + 2 v w + u \left(-2 v^2 + x\right)$

$-u v + w$

In[10]:=
```
DEN = Simplify[SymmetricReduction[Denominator[Together[F[[1]]]], SG, SP][[1]]]
```

Out[10]= $-x \left(-u v w + w^2 + u^2 x\right)$

(Check that the denominator is really a common denominator!)
Now for the chain rule: $\partial f_i / \partial g_j$

In[11]:=
```
DRV = Table[D[F[[i]], r_j] / (2 r_j), {i, 1, n}, {j, 0, n - 1}];
```

Multiply by $\partial g_j / \partial c_k$, giving the b coefficients [Eq.(A15)]

In[12]:=
```
B = Together[DRV.VMI];
```

Print in terms of symmetric polynomials.

In[13]:=
```
For[i = 1, i ≤ n, i++,
  For[j = i, j ≤ n, j++,
   NUMB[i, j] = Simplify[SymmetricReduction[Numerator[Together[B[[i]][[j]]]], SG, SP]];
   Print[NUMB[i, j][[1]]]
  ]
 ]
```

$$u^7 x^5 + u^6 w x^3 \left(w^2 - 8 v x\right) + u^5 x^2 \left(-3 v w^4 + 16 v^2 w^2 x - 4 v^3 x^2 + 5 w^2 x^2 + 4 v x^3\right) +$$
$$w^3 \left(w^6 - 3 v w^4 x - w^2 x^3 + v \left(v^2 - 2 x\right) x^3\right) - 3 u w^2 \left(v w^6 - w^4 x^2 + v^4 x^3 + x^5 - 3 v^2 x \left(w^4 + x^3\right)\right) +$$
$$u^4 w x \left(-12 v^3 w^2 x + 3 w^4 x + 3 v^4 x^2 - 23 v w^2 x^2 - 4 x^4 + v^2 \left(3 w^4 + 4 x^3\right)\right) +$$
$$u^2 w \left(3 w^6 x + 3 v^5 x^3 - 4 w^2 x^4 + 9 v x^2 \left(-2 w^4 + x^3\right) + 3 v^2 \left(w^6 + 2 w^2 x^3\right) - 3 v^3 \left(3 w^4 x + 4 x^4\right)\right) +$$
$$u^3 \left(-v^6 x^3 + 7 w^4 x^3 + x^6 - v^3 \left(w^6 + 8 w^2 x^3\right) + v^4 \left(3 w^4 x + 5 x^4\right) + v \left(-6 w^6 x + 4 w^2 x^4\right) + 3 v^2 \left(9 w^4 x^2 - 2 x^5\right)\right)$$

$$\begin{aligned}
&u^7 x^3 (2 w^2 - 3 v x) - u^6 w x^2 (6 v w^2 - 10 v^2 x + x^2) - w^3 (-v^3 w^2 x + 2 w^4 x - v^4 x^2 + x^4 + v^2 (w^4 + x^3)) + \\
&\quad u^5 x (-6 v^3 w^2 x + 6 w^4 x - 5 v^4 x^2 - 10 v w^2 x^2 + 2 v^2 (3 w^4 + x^3)) - \\
&\quad u^2 w (-3 v^5 w^2 x + 3 v^2 w^4 x - 3 v^6 x^2 - 12 v^3 w^2 x^2 + 3 x^2 (w^4 - x^3) + 3 v^4 (w^4 + 3 x^3) + v (6 w^6 + 5 w^2 x^3)) - \\
&\quad u^4 w (v^4 w^2 x - 6 v^5 x^2 - 12 v^2 w^2 x^2 + w^2 x^3 + 2 v^3 (w^4 - x^3) + 4 v (3 w^4 x - x^4)) + \\
&\quad u w^2 (2 w^6 - 3 v^4 w^2 x - 3 v^5 x^2 - 3 v^2 w^2 x^2 + 3 v^3 (w^4 + 2 x^3) + v (4 w^4 x + 3 x^4)) - u^3 (v^6 w^2 x - 6 w^6 x + v^7 x^2 + \\
&\qquad 15 v^4 w^2 x^2 + 5 w^2 x^4 - v^5 (w^4 + 4 x^3) - 2 v^2 (3 w^6 + 4 w^2 x^3) + v^3 (-2 w^4 x + 2 x^4) + v (3 w^4 x^2 + 2 x^5))
\end{aligned}$$

$$\begin{aligned}
&u^8 w x^3 - u^7 x^2 (3 v w^2 - 4 v^2 x + x^2) + u^6 w x (3 v^2 w^2 - 4 v^3 x + 3 w^2 x - 7 v x^2) - \\
&\quad w^3 (2 v w^4 - 2 v^2 w^2 x - 2 v^3 x^2 + w^2 x^2 + 3 v x^3) + u w^2 (-6 v^3 w^2 x - 6 v^4 x^2 + v w^2 x^2 - 3 x^4 + 3 v^2 (2 w^4 + 5 x^3)) + \\
&\quad u^4 w (-9 v^3 w^2 x + 5 v^4 x^2 - 12 v w^2 x^2 + 3 x (w^4 - 2 x^3) + v^2 (3 w^4 + 16 x^3)) + \\
&\quad u^3 (-2 v^5 w^2 x - 2 v^6 x^2 - 14 v^3 w^2 x^2 + x^2 (w^4 + x^3) + v^4 (2 w^4 + 9 x^3) - v (3 w^6 + 2 w^2 x^3) + 3 v^2 (5 w^4 x - 3 x^4)) + \\
&\quad u^5 (v^4 w^2 x + v^5 x^2 + 15 v^2 w^2 x^2 + w^2 x^3 - v^3 (w^4 + 12 x^3) + v (-6 w^4 x + 8 x^4)) + \\
&\quad u^2 w (w^6 + 6 v^4 w^2 x + 6 v^5 x^2 + 8 v^2 w^2 x^2 - 2 w^2 x^3 - 3 v^3 (2 w^4 + 7 x^3) + v (-7 w^4 x + 12 x^4))
\end{aligned}$$

$$\begin{aligned}
&-u^6 w x^3 + u^5 x^2 (3 v w^2 - 4 v^2 x + x^2) + u^4 w x (-3 v^2 w^2 + 4 v^3 x - 3 w^2 x + 5 v x^2) - w^3 (w^4 - v w^2 x + x^2 (-v^2 + x)) - \\
&\quad u w^2 (3 v^2 w^2 x + 3 v^3 x^2 + w^2 x^2 - 3 v (w^4 + 2 x^3)) + 3 u^2 w (v^3 w^2 x - w^4 x + v^4 x^2 + 2 v w^2 x^2 + x^4 - v^2 (w^4 + 3 x^3)) - \\
&\quad u^3 (v^4 w^2 x + v^5 x^2 + 9 v^2 w^2 x^2 + w^2 x^3 - v^3 (w^4 + 4 x^3) + v (-6 w^4 x + 3 x^4))
\end{aligned}$$

$$\begin{aligned}
&3 u^8 w x^3 - 3 u^7 x^2 (4 v w^2 + v^2 x + x^2) + \\
&\quad u^6 w x (12 v^2 w^2 + 16 v^3 x + 12 w^2 x + 15 v x^2) + w^3 (v^4 w^2 + v^5 x + 4 v^2 w^2 x + 2 w^2 x^2 - v x^3) - \\
&\quad u w^2 (3 v^5 w^2 + 3 v^6 x + 16 v^3 w^2 x + 8 w^4 x - 3 v^4 x^2 + 6 v w^2 x^2 - 3 x^4 + v^2 (4 w^4 - 6 x^3)) + \\
&\quad u^2 w (3 v^6 w^2 + 4 w^6 + 3 v^7 x + 27 v^4 w^2 x + 28 v w^4 x - 6 v^5 x^2 + 8 v^2 w^2 x^2 - 3 w^2 x^3 + 6 v^3 (2 w^4 - x^3)) - \\
&\quad u^5 (16 v^4 w^2 x + 24 v w^4 x + 5 v^5 x^2 + 36 v^2 w^2 x^2 + 17 w^2 x^3 + 4 v^3 (w^4 + x^3)) + \\
&\quad u^4 w (4 v^5 w^2 + 7 v^6 x + 44 v^3 w^2 x + 9 v^4 x^2 + 36 v w^2 x^2 + 4 v^2 (3 w^4 + 2 x^3) + 4 (3 w^4 x + x^4)) - \\
&\quad u^3 (v^7 w^2 + 12 v w^6 + v^8 x + 22 v^5 w^2 x - 3 v^6 x^2 + 8 v^3 w^2 x^2 + 16 w^4 x^2 + x^5 + v^4 (12 w^4 - x^3) + 2 v^2 (24 w^4 x + x^4))
\end{aligned}$$

$$\begin{aligned}
&-6 u^8 v w x^2 + u^9 x^3 + 2 u^7 x (3 v^2 w^2 + 2 v^3 x + 3 w^2 x) + \\
&\quad w^3 (2 v^3 w^2 + 2 v^4 x + 4 v w^2 x - v^2 x^2 - x^3) - u^6 w (2 v^3 w^2 + 5 v^4 x + 12 v w^2 x - 6 v^2 x^2 + 5 x^3) + \\
&\quad u w^2 (-6 v^4 w^2 - 4 v w^4 - 6 v^5 x - 16 v^2 w^2 x + 9 v^3 x^2 + 2 w^2 x^2 + 6 v x^3) + \\
&\quad u^4 w (v^4 w^2 - 6 v w^4 + 7 v^5 x + 9 v^2 w^2 x + 16 v^3 x^2 - w^2 x^2 + 10 v x^3) + \\
&\quad u^2 w (6 v^5 w^2 + 6 v^6 x + 29 v^3 w^2 x - 2 w^4 x - 15 v^4 x^2 - 4 v w^2 x^2 + 3 x^4 + v^2 (11 w^4 - 3 x^3)) + \\
&\quad u^5 (v^5 w^2 + v^6 x + 2 v^3 w^2 x - 11 v^4 x^2 - 9 v w^2 x^2 + v^2 (6 w^4 - 2 x^3) + 2 x (3 w^4 + x^3)) - \\
&\quad u^3 (2 v^6 w^2 - 2 w^6 + 2 v^7 x + 25 v^4 w^2 x - 7 v^5 x^2 + 3 v^2 w^2 x^2 + 5 w^2 x^3 + v^3 (9 w^4 + 2 x^3) + v (4 w^4 x + 3 x^4))
\end{aligned}$$

$$\begin{aligned}
&v^2 w^5 + v^3 w^3 x + 2 w^5 x + 6 u^6 v w x^2 - u^7 x^3 - 2 u^5 x (3 v^2 w^2 + 2 v^3 x + 3 w^2 x + v x^2) - \\
&\quad u w^2 (3 v^3 w^2 + 2 w^4 + 3 v^4 x + 8 v w^2 x - 3 v^2 x^2 - 3 x^3) + u^4 w (2 v^3 w^2 + 5 v^4 x + 12 v w^2 x + 6 v^2 x^2 + 4 x^3) + \\
&\quad u^2 w (3 v^4 w^2 + 6 v w^4 + 3 v^5 x + 15 v^2 w^2 x - 6 v^3 x^2 + w^2 x^2 - 3 v x^3) - \\
&\quad u^3 (v^5 w^2 + 6 v^2 w^4 + v^6 x + 14 v^3 w^2 x + 6 w^4 x - 3 v^4 x^2 + 3 v w^2 x^2 + x^4)
\end{aligned}$$

$$\begin{aligned}
&-3 u^9 v x^2 + 3 u^8 w x (v^2 + x) + 4 v w^3 (v w^2 + v^2 x - x^2) + 3 u^6 w (v^2 w^2 - 3 v^3 x + w^2 x - 6 v x^2) - \\
&\quad 3 u w^2 (4 v^3 w^2 + 4 v^4 x - 8 v^2 x^2 + x^3) - u^7 (v^3 w^2 + v^4 x + 6 v w^2 x - 15 v^2 x^2 + 3 x^3) + \\
&\quad u^2 w (12 v^4 w^2 - 4 v w^4 + 12 v^5 x + 8 v^2 w^2 x - 36 v^3 x^2 + 2 w^2 x^2 + 15 v x^3) + \\
&\quad u^4 (-12 v^3 w^3 + w^5 - 11 v w^3 x + 42 v^2 w x^2 - 7 w x^3) + \\
&\quad u^3 (-4 v^5 w^2 - 4 v^6 x - 12 v^3 w^2 x + 16 v^4 x^2 - 18 v w^2 x^2 + x^4 + v^2 (12 w^4 - 13 x^3)) + \\
&\quad u^5 (4 v^4 w^2 + 4 v^5 x + 21 v^2 w^2 x - 26 v^3 x^2 + 3 w^2 x^2 - 3 v (w^4 - 4 x^3))
\end{aligned}$$

$$3 u^7 v x^2 - 3 u^6 w x (v^2 + x) - 3 u v w^2 (2 v w^2 + 2 v^2 x - 3 x^2) + w^3 (2 v w^2 + 2 v^2 x - x^2) - 3 u^4 w (v^2 w^2 - v^3 x + w^2 x - 4 v x^2) + u^5 (v^3 w^2 + v^4 x + 6 v w^2 x - 9 v^2 x^2 + 2 x^3) + u^2 w (6 v^3 w^2 - w^4 + 6 v^4 x + 5 v w^2 x - 15 v^2 x^2 + 3 x^3) - u^3 (2 v^4 w^2 - 3 v w^4 + 2 v^5 x + 9 v^2 w^2 x - 7 v^3 x^2 + 3 w^2 x^2 + 4 v x^3)$$

$$w^5 + v w^3 x - 3 u^5 v x^2 + 3 u^4 w x (v^2 + x) - 3 u w^2 (v w^2 + v^2 x - x^2) + 3 u^2 w (v^2 w^2 + v^3 x + w^2 x - 2 v x^2) - u^3 (v^3 w^2 + v^4 x + 6 v w^2 x - 3 v^2 x^2 + x^3)$$

In[14]:=
```
DENB = SymmetricReduction[Denominator[Together[B[[1]][[1]]]], SG, SP][[1]]
```

Out[14]= $2 u^3 v^3 w^3 x^3 - 6 u^2 v^2 w^4 x^3 + 6 u v w^5 x^3 - 2 w^6 x^3 - 6 u^4 v^2 w^2 x^4 + 12 u^3 v w^3 x^4 - 6 u^2 w^4 x^4 + 6 u^5 v w x^5 - 6 u^4 w^2 x^5 - 2 u^6 x^6$

In[15]:=
```
Factor[DENB]
```

Out[15]= $-2 x^3 (-u v w + w^2 + u^2 x)^3$