A didactic derivation of the Boltzmann–Vlasov equation from multiple scattering using the Wigner function*

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Abstract

A derivation is given of the Boltzmann–Vlasov equation beginning from multiple scattering considerations. The motivation for the discussion, which is purely pedagogical in nature, is the current interest in understanding the origins of transport equations in terms of rigorous field-theory descriptions, or, as in this case, exact nonrelativistic formulations.

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*This article is dedicated to the memory of Eugene Wigner.

1 Introduction: deriving transport equations from rigorous formalisms

In 1932 Eugene Wigner introduced [1] the concept of what later came to be called the Wigner function, which allows a direct and precise mapping of quantum dynamics to transport equations and aids the study of the transition to the classical limit for these equations[1]. This concept has grown up as part of the standard tools of quantum theory. In recent years it has become of considerable importance in the study of relativistic heavy-ion collisions, both for the description of the dynamics of the phase in which quarks and gluons are confined in hadrons and for use in studying the deconfined phase of the quark-gluon plasma. The general line of attack in this context has been to use selected physics ingredients at the level of an exact formulation in terms of field theory or many-body dynamics and to exploit the Wigner function in order to derive the corresponding transport equation. It is then often possible to ignore detailed information about phases or fluctuations that is present in the full quantum theory but is not crucial for an approximate characterization of the system. Consequently the transport equation serves as a much simpler and much more transparent way of handling the basic physics in question.

This general approach raises the question, How well can transport methods represent the quantal situation? One can explore this issue in a formal derivation of the transport result beginning from the more fundamental theory—field theory or many-body theory, say—keeping track of approximations as one proceeds. It can also be studied by comparing numerical simulations of the results of the transport formalism with those of the fundamental theory. Both such methods have been used in recent years in the study of relativistic heavy-ion physics. Reviews pertaining to the various possible techniques can be found, for example, in refs. [2-6].

One of the natural techniques that can be used to generate transport equations for a many-body system—this time a nonrelativistic one—starts from the multiple-scattering series of Watson [7] or of Glauber [8] and derives from it the corresponding transport equation. This approach has the advantage that already from the start it is couched in the language of scattering, so that the progression to a transport formalism is conceptually quite natural. Of course, since the Watson multiple-scattering series can be derived directly [7] from the Schrödinger equation

[1]Wigner notes [1] that the Wigner function was actually found by Szilard and Wigner some years earlier for a purpose different from that of the 1932 paper.

through the Lippmann-Schwinger equation for N fixed potential scatterers, and since Glauber's series can in turn be derived from Watson's series [9, 10], such developments of transport theory are intrinsically rooted in basic nonrelativistic quantum theory. Notably, Thies [11] has systematically studied in full generality the derivation of the Wigner representation from Watson's multiple-scattering theory.

The purpose of the present brief note is to present a derivation of a transport equation using Wigner's function and multiple scattering together with the minimal set of assumptions needed in order to reach the simplest version of the transport equation, i.e., the analog of the classical Boltzmann–Vlasov equation. The motivation for presenting this is pedagogical: the complete treatment given by Thies [11] is quite intricate, so that there is an advantage to seeing also a more transparent development leading specifically to the well-known transport result. In some ways this makes it easier to focus on the nature of the approximations that are required in order to reach the Boltzmann-Vlasov equation.

2 Assumptions and formalism

We address a system in which a particle passes through a medium of N particles experiencing optical distortion on the way. In addition we consider one direct collision between the projectile and the ith particle within the medium; generalizations to consider n such collisions, where $n \ll N$, are possible. The target is taken to be a totally uncorrelated set of particles,

$$|\Phi_0(\mathbf{r}_1, \ldots, \mathbf{r}_N)|^2 = \rho_1(\mathbf{r}_1) \ldots \rho_N(\mathbf{r}_N), \tag{1}$$

where the complete lack of correlations implies that the densities for the individual particles are constants,

$$\rho_i(\mathbf{r}_i) = 1/\Omega, \tag{2}$$

where Ω is the volume containing the particles. It is clear that this extreme assumption is necessary in order to arrive at the "standard" Boltzmann–Vlasov equation at the lowest rung, where no correlations are entertained, of an anticipated hierarchy.

We now assume that the projectile possesses sufficiently high energy that an eikonal form for its wave function is a good approximation,

$$\begin{aligned} \phi_{\mathbf{k}}(\mathbf{r}; \mathbf{r}_i) &= \exp\left[i\mathbf{k}\cdot\mathbf{r}\right] \exp\left[-\frac{i}{v}\int_{-\infty}^{z} \mathcal{V}(\mathbf{b}, \zeta) d\zeta\right] \\ &\times \exp\left[-\frac{i}{v}\int_{-\infty}^{z} V_i(\mathbf{b} - \mathbf{b}_i, \zeta - z_i) d\zeta\right]. \end{aligned} \tag{3}$$

Here $\mathbf{k}$ is the momentum of the projectile, $\mathbf{v} = \mathbf{k}/m$ is its velocity with m its mass, $\mathcal{V}$ is the complex potential supplying the optical distortion, and V_i is the interaction between the projectile and the ith particle in the medium. The integrals are taken along the direction of $\mathbf{k}$, and the position vectors are taken as $\mathbf{r} = \{\mathbf{b}, z\}$, where $\mathbf{b}$ is the projection of $\mathbf{r}$ in a direction perpendicular to $\mathbf{k}$ and z is parallel to $\mathbf{k}$. Thus the ith particle in the target is located at $\mathbf{r}_i = \{\mathbf{b}_i, z_i\}$. The approximation of extreme high energy in eq. (3) is to be expected in order to arrive at a transport equation of classical form. This eikonal form is easily seen [7, 9, 12] to satisfy Schrödinger's equation approximately at high energy with a potential given by the optical potential $\mathcal{V}$ plus the force of the direct scattering V_i.

Wigner's function [1] for this static problem is defined as

$$\begin{aligned}
W(\mathbf{r};\,\mathbf{p}) \;\equiv\; & \int d\mathbf{r}_1 \cdots d\mathbf{r}_N \, \Phi_0^*(\mathbf{r}_1, \ldots, \mathbf{r}_N) \int d\boldsymbol{\eta} \exp\left[-i\mathbf{p}\cdot\boldsymbol{\eta}\right] \phi_{\mathbf{k}}(\mathbf{r} + \frac{1}{2}\boldsymbol{\eta}) \\
\times\; & \phi_{\mathbf{k}}^*(\mathbf{r} - \frac{1}{2}\boldsymbol{\eta}) \Phi_0(\mathbf{r}_1, \ldots, \mathbf{r}_N) \\
=\; & \int d\mathbf{r}_1 \cdots d\mathbf{r}_N \, |\Phi_0|^2 \int d\boldsymbol{\eta} \; \exp\left[i(\mathbf{k}-\mathbf{p})\cdot\boldsymbol{\eta}\right] \\
\times\; & \exp\left[-\frac{i}{v}\int_{-\infty}^{z+\frac{1}{2}\eta_\parallel} \mathcal{V}(\mathbf{b} + \frac{1}{2}\boldsymbol{\eta}_\perp, \zeta) d\zeta\right] \exp\left[\frac{i}{v}\int_{-\infty}^{z-\frac{1}{2}\eta_\parallel} \mathcal{V}^*(\mathbf{b} - \frac{1}{2}\boldsymbol{\eta}_\perp, \zeta) d\zeta\right] \\
\times\; & \exp\left[-\frac{i}{v}\int_{-\infty}^{z+\frac{1}{2}\eta_\parallel - z_i} V_i(\mathbf{b} - \mathbf{b}_i + \frac{1}{2}\boldsymbol{\eta}_\perp, \zeta) d\zeta\right] \\
\times\; & \exp\left[\frac{i}{v}\int_{-\infty}^{z-\frac{1}{2}\eta_\parallel - z_i} V_i^*(\mathbf{b} - \mathbf{b}_i - \frac{1}{2}\boldsymbol{\eta}_\perp, \zeta) d\zeta\right], \qquad (4)
\end{aligned}$$

where $\eta_\parallel$ and $\boldsymbol{\eta}_\perp$ are the components of $\boldsymbol{\eta}$ parallel and perpendicular to $\mathbf{k}$.

In order to generate a transport equation we consider the action of $v\,\partial/\partial z$ on the Wigner function of eq. (4). We divide the calculation into two parts: first we examine the action of the derivative on the optical part of the expression, containing $\mathcal{V}$, to obtain the drift part of the transport equation, and then we let it act on the direct collision pieces, containing V_i, to produce the collision term. The first calculation is very simple,

$$\begin{aligned}
v\frac{\partial}{\partial z} W(\mathbf{b}, z;\,\mathbf{p}) \;=\; & -i \int d\mathbf{r}_1 \cdots d\mathbf{r}_N \, |\Phi_0|^2 \int d\boldsymbol{\eta} \; \exp\left[i(\mathbf{k}-\mathbf{p})\cdot\boldsymbol{\eta}\right] \\
\times\; & \left[\mathcal{V}(\mathbf{b} + \frac{1}{2}\boldsymbol{\eta}_\perp, z + \frac{1}{2}\eta_\parallel) - \mathcal{V}^*(\mathbf{b} - \frac{1}{2}\boldsymbol{\eta}_\perp, z - \frac{1}{2}\eta_\parallel)\right] \exp[\cdots] \\
+\; & \text{collision terms} \\
\approx\; & -i \int d\mathbf{r}_1 \cdots d\mathbf{r}_N \, |\Phi_0|^2 \int d\boldsymbol{\eta} \; \exp\left[i(\mathbf{k}-\mathbf{p})\cdot\boldsymbol{\eta}\right]
\end{aligned}$$

$$
\begin{aligned}
&\times \quad [2\,i\,\mathrm{Im}\,\mathcal{V}(\mathbf{b},\mathrm{z}) + \boldsymbol{\eta}\cdot\boldsymbol{\nabla}\,\mathrm{Re}\,\mathcal{V}(\mathbf{b},\mathrm{z})]\exp[\cdots] + \text{collision terms} \\
&= \quad [2\,\mathrm{Im}\,\mathcal{V}(\mathbf{b},\mathrm{z}) + [\boldsymbol{\nabla}\,\mathrm{Re}\,\mathcal{V}(\mathbf{b},\mathrm{z})]\cdot\boldsymbol{\nabla}_{\mathbf{p}}]\; W(\mathbf{r},\mathbf{p}) \\
&+ \quad \text{collision terms}, \qquad (5)
\end{aligned}
$$

where we have have indicated only schematically the four exponentials of eq. (4) that appear unchanged in each of the subsequent expressions. In eq. (5) we have also used the usual approximation that the momentum $\mathbf{k}$ involved here is sufficiently high so that only small values of $\boldsymbol{\eta}$ enter in the integrals. Once again this is an approximation that is required for the classical limit to obtain. Equation (5) can be recast in the form

$$
\left[v\frac{\partial}{\partial z} - 2\,\mathrm{Im}\mathcal{V}(\mathbf{b},\mathrm{z}) - [\boldsymbol{\nabla}\,\mathrm{Re}\mathcal{V}]\cdot\boldsymbol{\nabla}_{\mathbf{p}}\right]\; W(\mathbf{r},\mathbf{p}) = \text{collision terms}. \qquad (6)
$$

We now turn to the collision term, omitting explicit reference to the contributions of the drift term that we have just calculated. We then have

$$
\begin{aligned}
v\frac{\partial}{\partial z}\,W(\mathbf{b},z;\mathbf{p}) \;&=\; -v\int d\mathbf{r}_i\,\rho(\mathbf{r}_i)\;\int d\boldsymbol{\eta}\,\exp[i(\mathbf{k}-\mathbf{p})\cdot\boldsymbol{\eta}] \\
&\times\; \frac{\partial}{\partial z_i}\Bigg\{\exp\left[-\frac{i}{v}\int_{-\infty}^{z+\frac{1}{2}\eta_\parallel - z_i} V_i(\mathbf{b}-\mathbf{b}_i+\frac{1}{2}\boldsymbol{\eta}_\perp,\zeta)d\zeta\right] \\
&\times\; \exp\left[\frac{i}{v}\int_{-\infty}^{z-\frac{1}{2}\eta_\parallel - z_i} V_i(\mathbf{b}-\mathbf{b}_i-\frac{1}{2}\boldsymbol{\eta}_\perp,\zeta)d\zeta\right]\Bigg\} \\
&\times\; \exp\left[-\frac{i}{v}\int_{-\infty}^{z+\frac{1}{2}\eta_\parallel} \mathcal{V}(\mathbf{b}+\frac{1}{2}\boldsymbol{\eta}_\perp,\zeta)d\zeta\right] \\
&\times\; \exp\left[\frac{i}{v}\int_{-\infty}^{z-\frac{1}{2}\eta_\parallel} \mathcal{V}^*(\mathbf{b}-\frac{1}{2}\boldsymbol{\eta}_\perp,\zeta)d\zeta\right], \qquad (7)
\end{aligned}
$$

where we have replaced the required derivative by z with one by z_i, as is easily done in the context of the collision part. The relevant limits of integration can be extended from $-\infty$ to ∞ if we assume that the longitudinal volume dimension $\Omega_\parallel$ is large relative to the domain of support of the scattering profiles. This is then leads to

$$
\begin{aligned}
v\frac{\partial}{\partial z}\,W(\mathbf{b},z;\mathbf{p}) \;&=\; -v\int d\boldsymbol{\eta}\,\exp[i(\mathbf{k}-\mathbf{p})\cdot\boldsymbol{\eta}]\int_\Omega d^2\mathbf{b}_i\,dz_i\,\frac{1}{\Omega} \\
&\times\; \frac{\partial}{\partial z_i}\Bigg\{\exp\left[-\frac{i}{v}\int_{-\infty}^{z+\frac{1}{2}\eta_\parallel - z_i} V_i(\mathbf{b}-\mathbf{b}_i+\frac{1}{2}\boldsymbol{\eta}_\perp,\zeta)d\zeta\right] \\
&\times\; \exp\left[\frac{i}{v}\int_{-\infty}^{z-\frac{1}{2}\eta_\parallel - z_i} V_i(\mathbf{b}-\mathbf{b}_i-\frac{1}{2}\boldsymbol{\eta}_\perp,\zeta)d\zeta\right]\Bigg\}\exp[\cdots] \qquad (8)
\end{aligned}
$$

where we use eq. (2) explicitly and again suppress the obvious "inert" exponentials, this time involving the optical potential. The integral over z_i is now trivial and we have

$$\begin{aligned} v\frac{\partial}{\partial z}\,W(\mathbf{b},z;\mathbf{p}) &= -v\int d\boldsymbol{\eta}\,\exp[i(\mathbf{k}-\mathbf{p})\cdot\boldsymbol{\eta}]\int_{\Omega_\perp} d^2\mathbf{b}_i\,\frac{1}{\Omega} \\ &\times \left\{1-\exp\left[-\frac{i}{v}\int_{-\infty}^{\infty} V_i(\mathbf{b}-\mathbf{b}_i+\frac{1}{2}\boldsymbol{\eta}_\perp,\zeta)d\zeta\right]\right. \\ &\times \left.\exp\left[\frac{i}{v}\int_{-\infty}^{\infty} V_i(\mathbf{b}-\mathbf{b}_i-\frac{1}{2}\boldsymbol{\eta}_\perp,\zeta)d\zeta\right]\right\}\exp[\cdots] \\ &= v\int d^2\boldsymbol{\eta}_\perp\int_{-\infty}^{\infty} d\eta_\parallel\int\frac{d\mathbf{p}'}{(2\pi)^3}\,\exp\left[i(\mathbf{p}'_\perp-\mathbf{p}_\perp)\cdot\boldsymbol{\eta}_\perp\right] \\ &\times \exp\left[i(p'_\parallel-p_\parallel)\eta_\parallel\right]\int_{\Omega_\perp} d^2\mathbf{b}_i\,\frac{1}{\Omega} \\ &\times \left\{[1+\Gamma_i(\mathbf{b}-\mathbf{b}_i+\frac{1}{2}\boldsymbol{\eta}_\perp)][1+\Gamma_i^*(\mathbf{b}-\mathbf{b}_i-\frac{1}{2}\boldsymbol{\eta}_\perp)]-1\right\} \\ &\times W(\mathbf{r};\,\mathbf{p}'), \end{aligned} \tag{9}$$

where the integration over the transverse variable $\mathbf{b}_i$ ranges over the transverse cross section $\Omega_\perp$ of the volume Ω, and we have defined the usual profile functions [8, 12],

$$\Gamma_i(\mathbf{B}) \equiv \exp\left[-\frac{i}{v}\int_{-\infty}^{\infty} V_i(\mathbf{B},\zeta)d\zeta\right]-1. \tag{10}$$

Then

$$\begin{aligned} v\frac{\partial}{\partial z}\,W(\mathbf{b},z\,;\mathbf{p}) &= 2\pi v\int\frac{d\mathbf{p}'}{(2\pi)^3}\,\delta(p'_\parallel-p_\parallel)\int d^2\boldsymbol{\eta}_\perp\int_{\Omega_\perp} d^2\mathbf{b}_i\,\frac{1}{\Omega} \\ &\times \left\{[1+\Gamma_i(\mathbf{b}-\mathbf{b}_i+\frac{1}{2}\boldsymbol{\eta}_\perp)][1+\Gamma_i^*(\mathbf{b}-\mathbf{b}_i-\frac{1}{2}\boldsymbol{\eta}_\perp)]-1\right\} \\ &\times \exp\left[i(\mathbf{p}'_\perp-\mathbf{p}_\perp)\cdot\boldsymbol{\eta}_\perp\right]\,W(\mathbf{r};\,\mathbf{p}') \\ &= 2\pi v\int\frac{d\mathbf{p}'}{(2\pi)^3}\,\delta(p'_\parallel-p_\parallel)\int d^2\mathbf{B}\,d^2\mathbf{B}' \\ &\times \frac{1}{\Omega}\left\{[1+\Gamma_i(\mathbf{B})][1+\Gamma_i^*(\mathbf{B}')]-1\right\} \\ &\times \exp\left[i(\mathbf{p}'_\perp-\mathbf{p}_\perp)\cdot(\mathbf{B}-\mathbf{B}')\right]\,W(\mathbf{r};\,\mathbf{p}'). \end{aligned} \tag{11}$$

The two-dimensional Fourier-transform integral over $\mathbf{B}$ and $\mathbf{B}'$ containing both profile functions immediately yields [8, 12] the scattering amplitudes at momentum transfer $\mathbf{p}'_\perp-\mathbf{p}_\perp$, assuming the transverse volume $\Omega_\perp$ is large relative to the domain of support of the scattering profile. Then

$$v\frac{\partial}{\partial z}\,W(\mathbf{b},z;\mathbf{p}) = 2\pi v\int\frac{d\mathbf{p}'}{(2\pi)^3}\,\delta(p'_\parallel-p_\parallel)\left(\frac{2\pi}{k}\right)^2\,\frac{d\sigma_i(\mathbf{p}'\to\mathbf{p})}{d\Omega}$$

$$\times \quad W(\mathbf{r};\, \mathbf{p}')\, \rho_i(\mathbf{r}), \tag{12}$$

where we have suppressed terms with transverse delta functions which do not contribute. We finally arrive at

$$v\frac{\partial}{\partial z}\, W(\mathbf{b}, z;\, \mathbf{p}) = \frac{1}{mk} \int d^2\mathbf{p}'_\perp\, \frac{d\sigma_i(\mathbf{p}' \to \mathbf{p})}{d\Omega}\, W(\mathbf{r};\, \mathbf{p}'_\perp, p_\parallel)\, \rho_i(\mathbf{r}) \tag{13}$$

for the collision term.

Putting the drift and collision pieces together we arrive at the Boltzmann-Vlasov transport equation

$$\begin{aligned} \Big[v\frac{\partial}{\partial z} - 2\,\mathrm{Im}\mathcal{V} \quad - \quad & \frac{\partial\, \mathrm{Re}\mathcal{V}}{\partial z}\, \frac{\partial}{\partial p_\parallel}\Big] W(\mathbf{b}, z;\, \mathbf{p}'_\perp, p_\parallel) = \\ = \quad & \int \frac{d^2\mathbf{p}'_\perp}{mk}\, \frac{d\sigma_i(\mathbf{p}' \to \mathbf{p})}{d\Omega}\, W(\mathbf{b}, z;\, \mathbf{p}'_\perp, p_\parallel)\rho_i(\mathbf{r}), \end{aligned} \tag{14}$$

which is the result arrived at by Thies [11] as the lowest-order limit of a complete, but necessarily much more intricate, development.[2]

3 Conclusions

Our result is wholly contained in eq. (14), and is the entirely expected Boltzmann–Vlasov equation with the role of the drift potential played by the background optical potential and explicit scattering appearing on the right-hand side as a collision term. This result is derived here in a relatively simple manner since we have been prepared to restrict ourselves from the very start to lowest-order features. In order to produce this classical transport equation our main assumptions have been (i) that the projectile enters the system at extremely high energy; (ii) that the scatterers within the bombarded system are completely without correlation; and (iii) that they are contained in a volume much larger than the support domain of the individual scatterings.

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[2]We note that with the present method it is unnecessary to assume [13] that the profile function always leads to forward contributions in the sense that $\Gamma(\mathbf{b} - \mathbf{b}_i,\, z - z_i) = \Gamma(\mathbf{b} - \mathbf{b}_i)\, \Theta(z - z_i)$.

References

[1] E. Wigner, Phys. Rev., 40 (1932) 749.

[2] J.M. Eisenberg, Y. Kluger, and A. Rosenhauer, in *The Nuclear Equation of State, Part B: QCD and the Formation of the Quark-Gluon Plasma,* W. Greiner and H. Stöcker, eds. (Plenum, New York, 1989) p. 501.

[3] H.-Th. Elze and U. Heinz, in *Quark-Gluon Plasma,* R.C. Hwa, ed. (World Scientific, Singapore, 1990) p. 117.

[4] S. Mrówczyński, *ibid.* p. 185.

[5] W. Cassing and U. Mosel, in *Prog. Part. Nucl. Phys. 25,* A. Faessler, ed. (Plenum, New York, 1991) p. 235.

[6] Y. Kluger, J.M. Eisenberg, and B. Svetitsky, Int. J. Mod. Phys. E, 2 (1993) 333.

[7] M.L. Goldberger and K.M. Watson, *Collision Theory* (Wiley, New York, 1964).

[8] R.J. Glauber, in *Lectures in Theoretical Physics 1* (Interscience, New York, 1959) p. 315.

[9] J.M. Eisenberg, Ann. Phys., 71 (1972) 542.

[10] J.M. Eisenberg, Nucl. Phys., A389, (1982) 595.

[11] M. Thies, Ann. Phys., 123 (1979) 411.

[12] J.M. Eisenberg and D.S. Koltun, *Theory of Meson Interactions with Nuclei* (Wiley-Interscience, New York, 1980).

[13] J. Hüfner, Ann. Phys., 115 (1978) 43.