## On variations of the action

Consider a harmonic oscillator,

$$L = \frac{1}{2}\dot{x}^2 - \frac{1}{2}\omega^2 x^2. \tag{1}$$

Consider paths with $x(0) = x(T) = 0$ where $T = 2\pi/\omega$ is the period of the oscillator,

$$S = \int_0^T L(x, \dot{x})\, dt. \tag{2}$$

Stationary paths are determined by

$$\frac{\delta S}{\delta x(t)} = -\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} + \frac{\partial L}{\partial x} = 0. \tag{3}$$

Inserting the Lagrangian, we find

$$\frac{\delta S}{\delta x(t)} = -\frac{d}{dt}\dot{x} - \omega^2 x, \tag{4}$$

so the equation of motion Eq.(3) is

$$\ddot{x} + \omega^2 x = 0, \tag{5}$$

which is solved by

$$x(t) = A \sin \omega t \tag{6}$$

for any value of $A$. This is a stationary point of the action and

$$L = \frac{1}{2}A^2\omega^2 \cos 2\omega t, \tag{7}$$

giving

$$S = \int_0^T L\, dt = 0. \tag{8}$$

Now to see whether these solutions are minima or maxima or what, we return to Eq.(4) and differentiate again,

$$\begin{aligned} \frac{\delta^2 S}{\delta x(t)\delta x(t')} &= \frac{\delta}{\delta x(t')}\left[-\ddot{x}(t) - \omega^2 x(t)\right] & (9) \\ &= -\delta''(t - t') - \omega^2 \delta(t - t'). & (10) \end{aligned}$$

This matrix has both positive and negative eigenvalues, which means that $\delta S$ can be either positive or negative, depending on the form of the variation $\delta x(t)$ around the solution (6). To be explicit,

$$S[x + \delta x] = S[x] + \int \frac{\delta S}{\delta x(t)}\delta x(t)\, dt + \frac{1}{2}\int\int \frac{\delta^2 S}{\delta x(t)\delta x(t')}\delta x(t)\delta x(t')\, dt\, dt' + \cdots. \tag{11}$$

The first term $S[x]$ is zero as above, and the first variation vanishes by the Lagrange equation. When we use Eq. (10), the second variation is seen to be

$$\frac{1}{2}\int dt\,\delta x(t)\left[-\delta\ddot{x}(t)-\omega^2\delta x(t)\right]. \tag{12}$$

Since $\delta x(t)$ must satisfy the boundary conditions, we can consider particular cases of the form $\delta x(t)=\epsilon\sin\frac{1}{2}n\omega t$. [These are eigenfunctions of the differential operator (10).] The second variation is then

$$\frac{1}{2}\left(\frac{1}{4}n^2-1\right)\omega^2\epsilon^2\int dt\,[\delta x(t)]^2. \tag{13}$$

This is negative for $n=1$; it is zero for $n=2$ (because it is a rescaling of the classical solution, and the action is independent of $A$); it is positive for $n>2$.

Thus the classical solution (6) is a saddle point. Note that the reason that the $n=1$ variation is unstable (i.e., a maximum at $\delta x=0$) is that we chose the variable $T$ to be an entire period of the oscillator. If we had chosen it to be a half-period, then the classical solution would have been stable against *all* variations except that of rescaling $A$. If we choose $T$ to be some other integer multiple of a half-period, say $\pi m/\omega$, then all variations with $n<m$ would represent instabilities.

*B. Svetitsky*
*School of Physics and Astronomy*
*Tel Aviv University*
*69978 Tel Aviv, Israel*

*http://julian.tau.ac.il/~bqs*

*July, 2002*