*B. Svetitsky, October 2021*

**TAYLOR EXPANSION OF THE COULOMB POTENTIAL**

The electrostatic potential of a charge distribution $\rho$ is given by

$$\phi(\boldsymbol{r}) = \int d^3r' \, \frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|} \ . \tag{1}$$

The Cartesian multipole expansion, valid at large distances $r$, can be derived by expanding $1/|\boldsymbol{r}-\boldsymbol{r}'|$ in powers of $\boldsymbol{r}'$. Treating $\boldsymbol{r}$ as a constant, we note first that

$$\left.\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\right|_{\boldsymbol{r}'=0} = \frac{1}{r} \ . \tag{2}$$

Continuing the Taylor expansion,

$$\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} = \left.\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\right|_{\boldsymbol{r}'=0} + r'_i \left(\frac{\partial}{\partial r'_i}\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\right)_{\boldsymbol{r}'=0} + \frac{1}{2} r'_i r'_j \left(\frac{\partial}{\partial r'_i}\frac{\partial}{\partial r'_j}\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\right)_{\boldsymbol{r}'=0} + \cdots \tag{3}$$

Now we note that for any function of the form $f(x-y)$,

$$\frac{\partial}{\partial y} f(x-y) = -\frac{\partial}{\partial x} f(x-y) \ , \tag{4}$$

and similarly,

$$\frac{\partial}{\partial r'_i}\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} = -\frac{\partial}{\partial r_i}\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} \ . \tag{5}$$

Using Eq. (5) for every derivative $\partial/\partial r'_i$ in Eq. (3), we obtain

$$\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} = \left.\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\right|_{\boldsymbol{r}'=0} - r'_i \left.\frac{\partial}{\partial r_i}\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\right|_{\boldsymbol{r}'=0} + \frac{1}{2} r'_i r'_j \left.\frac{\partial}{\partial r_i}\frac{\partial}{\partial r_j}\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\right|_{\boldsymbol{r}'=0} - \cdots \tag{6}$$

(note the minus sign), which gives finally

$$\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} = \frac{1}{r} - r'_i \frac{\partial}{\partial r_i}\left(\frac{1}{r}\right) + \frac{1}{2} r'_i r'_j \frac{\partial}{\partial r_i}\frac{\partial}{\partial r_j}\left(\frac{1}{r}\right) - \cdots \tag{7}$$

The integral in Eq. (1) thus becomes the multipole expansion,

$$\phi(\boldsymbol{r}) = \frac{q}{r} - \boldsymbol{p}\cdot\nabla\left(\frac{1}{r}\right) + \frac{1}{6} Q_{ij}\partial_i\partial_j\left(\frac{1}{r}\right) - \cdots \tag{8}$$

as derived in the lecture. (The 1/6 comes from the conventional definition of the quadrupole tensor $Q_{ij}$.)