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LONGITUDINAL PLASMA WAVES[1]

A plasma is a gas of electrons and ions. We assume the motion is non-relativistic; we ignore collisions between electrons and ions; and we ignore motion of the ions because they are heavy. If we are looking for longitudinal waves, $\boldsymbol{E} \parallel \boldsymbol{k}$, we start with

$$\nabla \times \boldsymbol{E} = 0, \tag{1}$$

which immediately tells us that $\partial \boldsymbol{B}/\partial t = 0$. For simplicity we assume further that $\boldsymbol{B} = 0$, and show that this gives a consistent solution of Maxwell's Equations.

We have Gauss's Law,

$$\nabla \cdot \boldsymbol{E} = 4\pi e n(\boldsymbol{r}), \tag{2}$$

where $n(\boldsymbol{r})$ is the density of electrons at $\boldsymbol{r}$. If indeed $\boldsymbol{B} = 0$, Ampère's Law gives

$$0 = \nabla \times \boldsymbol{B} = \frac{4\pi}{c}\boldsymbol{J} + \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t}. \tag{3}$$

The current is $\boldsymbol{J} = en\boldsymbol{v}$, where $\boldsymbol{v}(\boldsymbol{r})$ is the mean velocity of the electrons at $\boldsymbol{r}$.

Now we make an assumption that the plasma is nearly uniform and nearly static. This means that $n(\boldsymbol{r}) = n_0 + \delta n(\boldsymbol{r})$, where δn is small, and thus ∇n is small; also $\boldsymbol{v}$ is small. We work to first order in small quantities. Thus $n\boldsymbol{v} \to n_0\boldsymbol{v}$, giving the linearized current

$$\boldsymbol{J} = en_0\boldsymbol{v}. \tag{4}$$

Similarly, the continuity equation for $\boldsymbol{J}$,

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\boldsymbol{v}) = 0 \tag{5}$$

becomes

$$\frac{\partial n}{\partial t} = -n_0 \nabla \cdot \boldsymbol{v}. \tag{6}$$

Finally, we have the equation of motion of the electrons,

$$m\frac{d\boldsymbol{v}}{dt} = m\left[\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\right] = e\boldsymbol{E} - \frac{1}{n}\nabla p. \tag{7}$$

In Eq. (7) we used the "convective derivative," which takes note of the fact that $d\boldsymbol{v}/dt$ refers to a given electron, which moves during its acceleration. The term $(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}$ can, however, be neglected since it is of second order in the small quantity $\boldsymbol{v}$. The second term on the right-hand side is the force due to the pressure of the electron gas. The gradient $-\nabla p$ is a force per unit volume; to convert it into a force per particle we divide by n.

Now we are set up to derive a wave equation. We differentiate Ampère's Law (3) and use Eq. (4) to derive

$$\begin{aligned}\frac{1}{c}\frac{\partial^2 \boldsymbol{E}}{\partial t^2} &= -\frac{4\pi}{c}\frac{\partial \boldsymbol{J}}{\partial t} \\ &= -\frac{4\pi e n_0}{c}\frac{\partial \boldsymbol{v}}{\partial t}\end{aligned} \tag{8}$$

[1] Following Jackson (1st ed.) Sec. 10.9.

Using Eq. (7) (without the convective term),

$$\frac{1}{c}\frac{\partial^2 \boldsymbol{E}}{\partial t^2} = -\frac{4\pi e n_0}{mc}\left[e\boldsymbol{E} - \frac{1}{n_0}\nabla p\right] . \tag{9}$$

The gradient of p is proportional to the gradient of n via a thermodynamic derivative,

$$\nabla p = \left(\frac{\partial p}{\partial n}\right)_0 \nabla n, \tag{10}$$

where the derivative is an adiabatic derivative evaluated at $n = n_0$ (just as in the usual derivation of sound waves). The density n is related to $\boldsymbol{E}$ via Gauss's Law (2), so we have the final result

$$\frac{\partial^2 \boldsymbol{E}}{\partial t^2} + \frac{4\pi e^2 n_0}{m}\boldsymbol{E} - \frac{(\partial p/\partial n)_0}{m}\nabla(\nabla \cdot \boldsymbol{E}) = 0. \tag{11}$$

This is the wave equation.

We solve the wave equation as usual with a Fourier transform. In Fourier space, $\partial/\partial t \to -i\omega$ and $\nabla \to i\boldsymbol{k}$. Since $\boldsymbol{E}$ is longitudinal, it is parallel to $\boldsymbol{k}$ in Fourier space. Thus the wave equation becomes

$$\left[-\omega^2 + \omega_p^2 + \frac{(\partial p/\partial n)_0}{m}k^2\right]\boldsymbol{E}(\boldsymbol{k},\omega) = 0 \ , \tag{12}$$

where $\omega_p^2 = 4\pi e^2 n_0/m$ is called the plasma frequency. This translates into the dispersion relation,

$$\omega^2 = \omega_p^2 + \frac{(\partial p/\partial n)_0}{m}k^2 \ . \tag{13}$$

Plasma waves cannot propagate with a frequency $\omega < \omega_p$, since then k is imaginary. ω_p is the frequency of free oscillations of the plasma.