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INTEGRATION BY PARTS IN 3 DIMENSIONS

We show how to use Gauss' Theorem (the Divergence Theorem) to integrate by parts in three dimensions. In electrodynamics this method is used repeatedly in deriving static and dynamic multipole moments.

We recall that in one dimension, integration by parts comes from the Leibniz product rule for differentiation,

$$d(uv) = u\,dv + v\,du. \tag{1}$$

Then

$$\int_a^b u\,\frac{dv}{dx}\,dx = \int_a^b \frac{d(uv)}{dx}\,dx - \int_a^b v\,\frac{du}{dx}\,dx \tag{2}$$

and

$$\int_a^b \frac{d(uv)}{dx}\,dx = [uv]_a^b, \tag{3}$$

which vanishes if u or v vanishes at the boundary points. The three-dimensional method is similar.

We consider a current density $\boldsymbol{J}(\boldsymbol{r})$ that vanishes outside some finite region. Let the region R be larger than this region, meaning that its boundary surface ∂R is *outside* the region.

The divergence theorem states that

$$\int_R \nabla\cdot[\boldsymbol{J}(\boldsymbol{r})f(\boldsymbol{r})]\,dV = \oint_{\partial R} d\boldsymbol{S}\cdot\boldsymbol{J}(\boldsymbol{r})f(\boldsymbol{r}) \tag{4}$$

for any function $f(\boldsymbol{r})$, and this is *zero* since $\boldsymbol{J}$ is zero on the surface ∂R.

Now consider the tensor $J_i r_j$. We fix the second index j and calculate the divergence

$$\begin{aligned}
\nabla\cdot(\boldsymbol{J}r_j) &\equiv \partial_i(J_i r_j) && (5)\\
&= (\partial_i J_i)r_j + J_i(\partial_i r_j) && (6)\\
&= (\nabla\cdot\boldsymbol{J})r_j + J_j, && (7)
\end{aligned}$$

since $\partial_i r_j = \delta_{ij}$. This leads to the integration by parts of

$$\begin{aligned}
\int_R J_j\,dV &= \int_R \nabla\cdot(\boldsymbol{J}r_j)\,dV - \int_R (\nabla\cdot\boldsymbol{J})r_j\,dV && (8)\\
&= \oint_{\partial R} d\boldsymbol{S}\cdot(\boldsymbol{J}r_j) - \int_R (\nabla\cdot\boldsymbol{J})r_j\,dV && (9)\\
&= -\int_R (\nabla\cdot\boldsymbol{J})r_j\,dV, && (10)
\end{aligned}$$

which we can write as

$$\int_R \boldsymbol{J}\,dV = -\int_R (\nabla\cdot\boldsymbol{J})\boldsymbol{r}\,dV. \tag{11}$$

This is used in showing that there is no static monopole magnetic moment since for steady currents $\nabla\cdot\boldsymbol{J} = 0$. We also use it in deriving electric dipole (E1) radiation, where $\nabla\cdot\boldsymbol{J} = -\partial\rho/\partial t = i\omega\rho$.

A similar derivation involves the third-rank tensor $J_i r_j r_k$. Fixing j and k, we have

$$\begin{aligned}
\nabla \cdot (\boldsymbol{J} r_j r_k) &\equiv \partial_i (J_i r_j r_k) && (12)\\
&= (\partial_i J_i) r_j r_k + J_i (\partial_i r_j) r_k + J_i r_j (\partial_i r_k) && (13)\\
&= (\nabla \cdot \boldsymbol{J}) r_j r_k + J_j r_k + J_k r_j. && (14)
\end{aligned}$$

Then we can integrate by parts as follows,

$$\begin{aligned}
\int_R (J_j r_k + J_k r_j)\, dV &= \int_R \nabla \cdot (\boldsymbol{J} r_j r_k) - \int_R (\nabla \cdot \boldsymbol{J}) r_j r_k, dV && (15)\\
&= \oint_{\partial R} d\boldsymbol{S} \cdot (\boldsymbol{J} r_j r_k) - \int_R (\nabla \cdot \boldsymbol{J}) r_j r_k, dV && (16)\\
&= -\int_R (\nabla \cdot \boldsymbol{J}) r_j r_k \, dV. && (17)
\end{aligned}$$

We use this in deriving electric quadrupole (E2) radiation.