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MAGNETIC MOMENT OF A CURRENT LOOP

The Cartesian multipole expansion for the magnetic vector potential has the dipole term for its first nontrivial term. Consider a current loop. The current I flows around a closed curve $\mathcal{C}$, defined by the vector function $\boldsymbol{x}(\ell)$ of the parameter ℓ along its length (see the notes on charge and current density). Then the magnetic dipole moment is

$$\boldsymbol{\mu} = \frac{I}{2c} \oint_{\mathcal{C}} \boldsymbol{x}(\ell) \times d\boldsymbol{\ell}. \tag{1}$$

We can turn this into an area integral as follows. Since $\boldsymbol{x}$ is just the location of each loop element, we can regard it as the field $\boldsymbol{r}$ evaluated on the loop. $\boldsymbol{r}$ is of course defined everywhere in space; another way of writing Eq. (1) is

$$\boldsymbol{\mu} = \frac{I}{2c} \oint_{\mathcal{C}} \boldsymbol{r} \times d\boldsymbol{\ell}. \tag{2}$$

If we write this in terms of components, we have

$$\mu_i = \frac{I}{2c} \oint_{\mathcal{C}} \epsilon_{ijk}\, r_j\, d\ell_k. \tag{3}$$

Now let's keep i fixed, and define a new vector field $\boldsymbol{F}$ as

$$F_k = \epsilon_{ijk} r_j, \tag{4}$$

so that

$$\mu_i = \frac{I}{2c} \oint_{\mathcal{C}} \boldsymbol{F} \cdot d\boldsymbol{\ell}. \tag{5}$$

(Remember i is fixed, and it is hiding inside the definition of $\boldsymbol{F}$.) Equation (5) is the circulation of $\boldsymbol{F}$ around the loop; by Stokes' Theorem it is equal to

$$\mu_i = \frac{I}{2c} \int_{\mathcal{S}} (\nabla \times \boldsymbol{F}) \cdot d\boldsymbol{S}, \tag{6}$$

where $\mathcal{S}$ is *any* surface bounded by the closed curve $\mathcal{C}$. Let's evaluate $\nabla \times \boldsymbol{F}$:

$$\begin{aligned} (\nabla \times \boldsymbol{F})_l &= \epsilon_{lmn} \partial_m \epsilon_{ijn} r_j = \epsilon_{lmn} \epsilon_{ijn} \delta_{jm} \\ &= \epsilon_{lmn} \epsilon_{imn} = 2\delta_{il}, \end{aligned} \tag{7}$$

so

$$\mu_i = \frac{I}{c} \int_{\mathcal{S}} dS_i, \tag{8}$$

which means

$$\boldsymbol{\mu} = \frac{I}{c} \int_{\mathcal{S}} d\boldsymbol{S} = \frac{I}{c} \mathcal{A}_{\hat{\boldsymbol{n}}}(\mathcal{S}) \hat{\boldsymbol{n}}. \tag{9}$$

Remember that $\mathcal{C}$ is not necessarily a planar curve, and also that $\mathcal{S}$ is *any* surface bounded by $\mathcal{C}$. In Eq. (9), then, $\hat{\boldsymbol{n}}$ is some mean of the normal vectors to the elements $d\boldsymbol{S}$, and $\mathcal{A}_{\hat{\boldsymbol{n}}}(\mathcal{S})$

is the area of the projection of $\mathcal{S}$ onto a plane normal to $\hat{\boldsymbol{n}}$.[1] We got here from Eq. (1), which is just a line integral around the curve $\mathcal{C}$, so we already know that the result (9) is independent of our choice of the surface $\mathcal{S}$!

If we make $\mathcal{C}$ a planar curve, then it is clear that

$$\boldsymbol{\mu} = \frac{I}{c}\mathcal{A}(\mathcal{S})\hat{\boldsymbol{n}}. \tag{10}$$

Here $\mathcal{S}$ can be chosen to be the planar surface enclosed by $\mathcal{C}$; then $\mathcal{A}(\mathcal{S})$ is just the area of this surface and $\hat{\boldsymbol{n}}$ is the normal to the plane.

[1] If you're confused by this, do the exercise of evaluating $\int_{\mathcal{S}} d\boldsymbol{S}$ on a hemisphere.