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LONGITUDINAL AND TRANSVERSE FIELDS

Problem: Given a vector function $\boldsymbol{F}(\boldsymbol{r})$, find two vector functions $\boldsymbol{F}_L(\boldsymbol{r})$ and $\boldsymbol{F}_T(\boldsymbol{r})$ such that

$$\boldsymbol{F} = \boldsymbol{F}_L + \boldsymbol{F}_T \tag{1}$$

$$\nabla \cdot \boldsymbol{F}_T = 0 \tag{2}$$

$$\nabla \times \boldsymbol{F}_L = 0. \tag{3}$$

We assume the fields go to zero at infinity fast enough for whatever we will say here.

First question: Is there a unique answer? After all, given a solution $\boldsymbol{F}_L, \boldsymbol{F}_T$ we can find another solution $\boldsymbol{F}'_L, \boldsymbol{F}'_T$ by writing, for any function $\Lambda(\boldsymbol{r})$,

$$\boldsymbol{F}'_L = \boldsymbol{F}_L + \nabla\Lambda \tag{4}$$

$$\boldsymbol{F}'_T = \boldsymbol{F}_T - \nabla\Lambda. \tag{5}$$

Then $\nabla \times \boldsymbol{F}'_L$ is still zero, and $\nabla \cdot \boldsymbol{F}'_T$ will be zero if we demand $\nabla^2\Lambda = 0$. But this means that $\Lambda(\boldsymbol{r})$ must satisfy Laplace's equation. If all fields go to zero at infinity, then Λ must go to a constant at infinity. When we solve Laplace's equation with this boundary condition the only solution is $\Lambda =$ constant, which means that $\boldsymbol{F}_L, \boldsymbol{F}_T$ are unique.

To solve the problem: We assume that any $\boldsymbol{F}_L$ satisfying Eq. (3) can be written as a gradient,

$$\boldsymbol{F}_L = -\nabla\phi. \tag{6}$$

Then, since $\nabla \cdot \boldsymbol{F}_T = 0$,

$$\nabla \cdot \boldsymbol{F} = \nabla \cdot \boldsymbol{F}_L = -\nabla^2\phi. \tag{7}$$

But this is just Poisson's equation for ϕ, with the solution

$$\phi(\boldsymbol{r}) = \frac{1}{4\pi}\int d^3r' \, \frac{\nabla' \cdot \boldsymbol{F}(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|}, \tag{8}$$

so

$$\boxed{\boldsymbol{F}_L(\boldsymbol{r}) = -\nabla \left[\frac{1}{4\pi}\int d^3r' \, \frac{\nabla' \cdot \boldsymbol{F}(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|}\right]} \tag{9}$$

To find $\boldsymbol{F}_T$, we suppose that it can be written as a curl,

$$\boldsymbol{F}_T = \nabla \times \boldsymbol{A}. \tag{10}$$

There is a gauge freedom here, $\boldsymbol{A} \to \boldsymbol{A} + \nabla\Lambda$; we use it to impose a condition on $\boldsymbol{A}$, that $\nabla \cdot \boldsymbol{A} = 0$. Since $\nabla \times \boldsymbol{F}_L = 0$,

$$\nabla \times \boldsymbol{F} = \nabla \times \boldsymbol{F}_T = \nabla \times \nabla \times \boldsymbol{A} = -\nabla^2\boldsymbol{A}. \tag{11}$$

This is again Poisson's equation, with the solution

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{1}{4\pi}\int d^3r' \, \frac{\nabla' \times \boldsymbol{F}(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|}. \tag{12}$$

Thus

$$\boxed{\boldsymbol{F}_T(\boldsymbol{r}) = \nabla \times \left[\frac{1}{4\pi} \int d^3r' \, \frac{\nabla' \times \boldsymbol{F}(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} \right]} \tag{13}$$

The various assumptions made along the way are justified by verifying that our solution indeed satisfied Eqs. (1)–(3); uniqueness does the rest. Note:

- $\boldsymbol{F}_L$ is a gradient, and is determined by $\nabla \cdot \boldsymbol{F}$.
- $\boldsymbol{F}_T$ is a curl, and is determined by $\nabla \times \boldsymbol{F}$.
- If $\nabla \cdot \boldsymbol{F} = 0$ then $\boldsymbol{F} = \boldsymbol{F}_T$, which is a curl.
- If $\nabla \times \boldsymbol{F} = 0$, then $\boldsymbol{F} = \boldsymbol{F}_L$, which is a gradient.

We can solve the problem another way, more directly, using Fourier integrals. We write $\boldsymbol{F}$ as a Fourier integral,

$$\boldsymbol{F}(\boldsymbol{r}) = \frac{1}{(2\pi)^{3/2}} \int d^3k \, \boldsymbol{f}(\boldsymbol{k}) \, e^{i\boldsymbol{k}\cdot\boldsymbol{r}}, \tag{14}$$

and the same for $\boldsymbol{F}_L, \boldsymbol{F}_T$ in terms of $\boldsymbol{f}_L, \boldsymbol{f}_T$. Clearly $\boldsymbol{f}(\boldsymbol{k}) = \boldsymbol{f}_L(\boldsymbol{k}) + \boldsymbol{f}_T(\boldsymbol{k})$.

The divergence and curl of $\boldsymbol{F}_{T,L}$ are

$$\nabla \cdot \boldsymbol{F}_T(\boldsymbol{r}) = \frac{1}{(2\pi)^{3/2}} \int d^3k \, (i\boldsymbol{k} \cdot \boldsymbol{f}_T(\boldsymbol{k})) \, e^{i\boldsymbol{k}\cdot\boldsymbol{r}} \tag{15}$$

$$\nabla \times \boldsymbol{F}_L(\boldsymbol{r}) = \frac{1}{(2\pi)^{3/2}} \int d^3k \, (i\boldsymbol{k} \times \boldsymbol{f}_L(\boldsymbol{k})) \, e^{i\boldsymbol{k}\cdot\boldsymbol{r}}. \tag{16}$$

This shows that if we demand, for all $\boldsymbol{k}$, that

$$\boldsymbol{k} \cdot \boldsymbol{f}_T(\boldsymbol{k}) = 0 \tag{17}$$

$$\boldsymbol{k} \times \boldsymbol{f}_L(\boldsymbol{k}) = 0, \tag{18}$$

then we will have the solution. These equations mean that, for each value of $\boldsymbol{k}$, we have to break up the vector $\boldsymbol{f}(\boldsymbol{k})$ into components parallel and perpendicular to $\boldsymbol{k}$.

Equation (18) means that $\boldsymbol{f}_L(\boldsymbol{k})$ is parallel to $\boldsymbol{k}$. The solution is

$$\boldsymbol{f}_L(\boldsymbol{k}) = \frac{(\boldsymbol{k} \cdot \boldsymbol{f})}{k^2} \boldsymbol{k}, \tag{19}$$

or, in components,

$$f_{Li} = \frac{k_i k_j}{k^2} f_j. \tag{20}$$

$\boldsymbol{f}_T(\boldsymbol{k})$, on the other hand, is perpendicular to $\boldsymbol{k}$. The solution is

$$\boldsymbol{f}_T(\boldsymbol{k}) = -\frac{\boldsymbol{k} \times (\boldsymbol{k} \times \boldsymbol{f})}{k^2} \tag{21}$$

$$= \boldsymbol{f} - \frac{(\boldsymbol{k} \cdot \boldsymbol{f})}{k^2} \boldsymbol{k}. \tag{22}$$

In components,

$$f_{Ti} = \left(\delta_{ij} - \frac{k_i k_j}{k^2} \right) f_j. \tag{23}$$

The forms (19) and (21) will be most useful to us. Both are ambiguous as $\boldsymbol{k} \to 0$. We can remove this ambiguity if we replace k^2 by $k^2 + i\epsilon$ in the denominators; then

$$\lim_{\boldsymbol{k}\to 0} \boldsymbol{f}_{L,T}(\boldsymbol{k}) = 0, \tag{24}$$

which is appropriate since $\boldsymbol{f}_{L,T}(\boldsymbol{k} = 0)$ governs the constant component of $\boldsymbol{F}_{L,T}(\boldsymbol{r})$. This is zero by assumption $[\boldsymbol{F}_{L,T}(\infty) \to 0]$.

Now we return to $\boldsymbol{r}$ space. Transforming back, we have

$$\begin{aligned} \boldsymbol{F}_L(\boldsymbol{r}) &= \frac{1}{(2\pi)^{3/2}} \int d^3k\, \boldsymbol{k} \frac{1}{k^2 + i\epsilon} \boldsymbol{k} \cdot \boldsymbol{f}(\boldsymbol{k})\, e^{i\boldsymbol{k}\cdot\boldsymbol{r}} & (25) \\ &= (-i\nabla)(-i\nabla\cdot)\boldsymbol{G}(\boldsymbol{r}), & (26) \end{aligned}$$

where we define

$$\boldsymbol{G}(\boldsymbol{r}) = \frac{1}{(2\pi)^{3/2}} \int d^3k\, \frac{1}{k^2 + i\epsilon} \boldsymbol{f}(\boldsymbol{k})\, e^{i\boldsymbol{k}\cdot\boldsymbol{r}}. \tag{27}$$

What is $\boldsymbol{G}(\boldsymbol{r})$? It is easy to see that

$$\begin{aligned} \nabla^2 \boldsymbol{G} &= \frac{1}{(2\pi)^{3/2}} \int d^3k\, \frac{-k^2}{k^2 + i\epsilon} \boldsymbol{f}(\boldsymbol{k})\, e^{i\boldsymbol{k}\cdot\boldsymbol{r}} & (28) \\ &= -\boldsymbol{F}(\boldsymbol{r}). & (29) \end{aligned}$$

This is Poisson's equation; in view of the boundary conditions, the solution is

$$\boldsymbol{G}(\boldsymbol{r}) = \frac{1}{4\pi} \int d^3r'\, \frac{\boldsymbol{F}(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|}. \tag{30}$$

Our result is

$$\begin{aligned} \boldsymbol{F}_L(\boldsymbol{r}) &= -\nabla \left[\frac{1}{4\pi} \int d^3r'\, \nabla \cdot \frac{\boldsymbol{F}(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} \right], & (31) \\ &= -\nabla \left[\frac{1}{4\pi} \int d^3r'\, \boldsymbol{F}(\boldsymbol{r}') \cdot \nabla \frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|} \right]. & (32) \end{aligned}$$

We can show that this is the same as our earlier result. Note that

$$\nabla \frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|} = -\nabla' \frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|}\ , \tag{33}$$

where ∇' means the gradient with respect to $\boldsymbol{r}'$. We insert this into Eq. (32) and integrate by parts, which means we use

$$\boldsymbol{F}(\boldsymbol{r}') \cdot \nabla' \frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|} = \nabla' \cdot \left(\boldsymbol{F}(\boldsymbol{r}') \frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|} \right) - \left(\nabla' \cdot \boldsymbol{F}(\boldsymbol{r}') \right) \frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|}. \tag{34}$$

When Eq. (34) is integrated over the volume, the first term on the RHS turns into a surface integral at infinity,

$$\int d^3r'\, \nabla' \cdot \left(\boldsymbol{F}(\boldsymbol{r}') \frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|} \right) = \iint d\boldsymbol{S} \cdot \boldsymbol{F}(\boldsymbol{r}') \frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|} \to 0. \tag{35}$$

We arrive at the result

$$\boldsymbol{F}_L(\boldsymbol{r}) = -\nabla \left[\frac{1}{4\pi} \int d^3r' \, \frac{\nabla' \cdot \boldsymbol{F}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|} \right], \qquad QED. \tag{36}$$

For the transverse component we have similarly

$$\begin{aligned} \boldsymbol{F}_T(\boldsymbol{r}) &= \nabla \times \left[\frac{1}{4\pi} \int d^3r' \, \nabla \times \frac{\boldsymbol{F}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|} \right] && (37) \\ &= \nabla \times \left[\frac{1}{4\pi} \int d^3r' \left(\nabla \frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} \right) \times \boldsymbol{F}(\boldsymbol{r}') \right], && (38) \end{aligned}$$

which we can transform by similar steps to

$$\boldsymbol{F}_T(\boldsymbol{r}) = \nabla \times \left[\frac{1}{4\pi} \int d^3r' \, \frac{\nabla' \times \boldsymbol{F}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|} \right], \tag{39}$$

as above.