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WHEN IS A CLOSED FORM EXACT?

This is a pompous title for the question: If we have a vector field satisfying

$$\nabla \times \boldsymbol{F} = 0, \tag{1}$$

is it always true that it can be expressed as a gradient,

$$\boldsymbol{F} = \nabla\phi \ ? \tag{2}$$

In infinite three-dimensional space, the answer is *yes.* For proof, we can just construct the potential $\phi(\boldsymbol{r})$. See the notes on *Longitudinal and transverse fields.* There it is shown how to decompose a field $\boldsymbol{F}(\boldsymbol{r})$ into two pieces, so that

$$\boldsymbol{F} = \boldsymbol{F}_T + \boldsymbol{F}_L \tag{3}$$

and

$$\nabla \cdot \boldsymbol{F}_T = 0, \quad \nabla \times \boldsymbol{F}_L = 0. \tag{4}$$

The solution is

$$\boldsymbol{F}_T = \nabla \times \boldsymbol{A}, \quad \boldsymbol{F}_L = \nabla\phi, \tag{5}$$

and explicit expressions for $\boldsymbol{A}$ and ϕ are given. In particular, if $\nabla \times \boldsymbol{F} = 0$ then $\boldsymbol{F}_T = 0$.

Now for a counterexample. Consider three-dimensional space with a hole cut out of it. In cylindrical coordinates (r, θ, z) we allow only $r > R$; the region $r < R$ is excluded. The important thing is that this "space" is not *simply connected.* The precise meaning of this is that there exist closed curves that *cannot* be continuously shrunk to a point (namely, any curve that encloses the hole).

Now take the vector field

$$\boldsymbol{F} = \frac{1}{r}\hat{\boldsymbol{\theta}}. \tag{6}$$

Straightforward calculation shows that $\nabla \times \boldsymbol{F} = 0$. Or we can be less straightforward: This $\boldsymbol{F}$ is just the magnetic field of a straight wire; outside the wire, Ampère's Law says that $\nabla \times \boldsymbol{B} = 0$. (In fact $\nabla \times \boldsymbol{B}$ is only nonzero *on* the wire, but here the wire is in the hole, which is outside the space.)

Can $\boldsymbol{F}(\boldsymbol{r})$ be written as a gradient? Suppose it can, and write $\boldsymbol{F} = \nabla\phi$. Now calculate the integral of $\boldsymbol{F}$ along any circular path around the hole,

$$\mathcal{I} = \oint \boldsymbol{F} \cdot d\boldsymbol{l} = \int_0^{2\pi} \frac{1}{r}\, r\, d\theta = 2\pi. \tag{7}$$

But if $\boldsymbol{F} = \nabla\phi$, then

$$\oint \boldsymbol{F} \cdot d\boldsymbol{l} = \oint (\nabla\phi) \cdot d\boldsymbol{l} = \phi(2\pi) - \phi(0), \tag{8}$$

since for a closed loop the starting point and end point are the same. This should be zero if ϕ is a well-defined, single-valued potential! So we have reached a contradiction: $\boldsymbol{F}$ cannot be represented as the gradient of a single-valued potential. In physics terms: $\boldsymbol{F}$ is not a conservative force field even though its curl is zero.

Note also that Stokes' Theorem doesn't work. If

$$\oint \boldsymbol{F} \cdot d\boldsymbol{l} = \int\int (\nabla \times \boldsymbol{F}) \cdot d\boldsymbol{S}, \tag{9}$$

then in this case the RHS can't be calculated because of the hole.

A related question: If we are given $\boldsymbol{F}(\boldsymbol{r})$ such that $\nabla \cdot \boldsymbol{F} = 0$, can $\boldsymbol{F}$ be written as a curl? Again, in infinite volume the answer is yes, by the same argument as above. But a counterexample can be constructed in three-dimensional space with a *spherical* hole. Take

$$\boldsymbol{F} = \frac{1}{r^2}\hat{\boldsymbol{r}}, \tag{10}$$

in spherical coordinates, for $r > R$. This is a Coulomb field for a charge at the center of the hole, so $\nabla \cdot \boldsymbol{F} = 0$. Suppose we can write $\boldsymbol{F} = \nabla \times \boldsymbol{A}$. Then take any sphere enclosing the hole, and calculate the flux

$$\Phi = \int\int \boldsymbol{F} \cdot d\boldsymbol{S} = \int\int \frac{1}{r^2}\, r^2\, d\Omega = 4\pi. \tag{11}$$

But

$$\int\int \boldsymbol{F} \cdot d\boldsymbol{S} = \int\int (\nabla \times \boldsymbol{A}) \cdot d\boldsymbol{S} = \oint \boldsymbol{A} \cdot d\boldsymbol{l}, \tag{12}$$

where the final integral is a line integral along the boundary of the surface. But the sphere is a closed surface, *with no boundary!* Thus the last integral is zero, and we have reached a contradiction. It should be clear that we have taken advantage of the inapplicability of Gauss' Theorem here, just as we encountered the loss of Stokes' Theorem above.

Incidentally this shows that a magnetic monopole can't be described by a vector potential, even if we exclude the monopole itself and try to deal with the rest of space, where $\nabla \cdot \boldsymbol{B} = 0$.