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CHARGE AND CURRENT DENSITY ON A LOOP

A curve in space is specified by a vector function $\boldsymbol{x}(\ell)$ of a single parameter ℓ that tells you where you are on the loop. ℓ can be anything, but it is most convenient to use the curve itself for defining ℓ: We make it the distance one has travelled from the start of the curve. So

- $\boldsymbol{x}(\ell)$ is where we are after going a distance ℓ along the curve.
- $\boldsymbol{x}(0)$ is the beginning of the curve.
- $\boldsymbol{x}(L)$ is the end of the curve, whose total length is L.

If the curve is charged with linear density λ [*charge per unit length*] then an element of length $d\ell$ carries charge in the amount $\lambda\, d\ell$. If this element is located at $\boldsymbol{x}(\ell)$ then the charge density distribution due to this element is

$$\rho_\ell(\boldsymbol{r}) = \lambda\, d\ell\, \delta(\boldsymbol{r} - \boldsymbol{x}(\ell)), \tag{1}$$

that is, it is zero unles you sit on the element. But all the elements of the curve contribute to $\rho(\boldsymbol{r})$, each contributing its δ function:

$$\rho(\boldsymbol{r}) = \sum_\ell \rho_\ell(\boldsymbol{r}). \tag{2}$$

As $d\ell \to 0$ the sum becomes an integral,

$$\boxed{\rho(\boldsymbol{r}) = \int_0^L \lambda\, d\ell\, \delta(\boldsymbol{r} - \boldsymbol{x}(\ell))} \tag{3}$$

This is the charge density distribution in space specified by the charged curve.

Now consider a closed curve, so $\boldsymbol{x}(L) = \boldsymbol{x}(0)$. Let the charge flow along the curve with speed v. At $\boldsymbol{x}(\ell)$ the direction of the velocity is along the tangent to the curve, which is the unit vector

$$\hat{\boldsymbol{t}}\Big|_\ell = \frac{d\boldsymbol{x}(\ell)}{d\ell}. \tag{4}$$

In other words, $\boldsymbol{v}(\ell) = v\, \hat{\boldsymbol{t}}\Big|_\ell$. The current density is $\boldsymbol{J}_l(\boldsymbol{r}) = \rho_\ell(\boldsymbol{r})\boldsymbol{v}$, which is

$$\boldsymbol{J}_l(\boldsymbol{r}) = \lambda\, d\ell\, \delta(\boldsymbol{r} - \boldsymbol{x}(\ell))\, (v\hat{\boldsymbol{t}}) \tag{5}$$

$$= \lambda v\, d\boldsymbol{l}\, \delta(\boldsymbol{r} - \boldsymbol{x}(\ell)). \tag{6}$$

Adding up all the elements of the curve,

$$\boldsymbol{J}(\boldsymbol{r}) = \sum_\ell \boldsymbol{J}_\ell(\boldsymbol{r}) = \lambda v \oint d\boldsymbol{l}\, \delta(\boldsymbol{r} - \boldsymbol{x}(\ell)). \tag{7}$$

Since $\lambda v = I$, the current in the loop, we have

$$\boxed{\boldsymbol{J}(\boldsymbol{r}) = I \oint d\boldsymbol{l}\, \delta(\boldsymbol{r} - \boldsymbol{x}(\ell))} \tag{8}$$